Summary

An auction is a mechanism to buy (sell) a product or service, where potential sellers (buyers) have to submit bids, which fully determine the winner of the auction and the price he or she has to pay (or receives). Famous examples of auctions are the art-auctions in auction houses like Sotheby's and Christie's, which are the two largest auction houses at this moment. However, auctions do not always need to take place in such an auction house, but can actually take place anywhere. A virtual place to auction all kind of items, which is rapidly growing during recent years, is Ebay.

There are a lot of different rules that can be applied in a certain auction. For example, the rules describing when and how often bidders may submit bids, when the auction ends, how the winner will be determined and which amount every participant has to pay, can be totally different between one auction and another. Although all these characteristics may be different, most of the best known auction types have in common that bids involve only a certain price for which the bidder wants to buy the product or deliver the service. Stating a price is enough in some situations, but it is not always the only interesting aspect. Auctions in which not only the price, but also other factors are important, are known as multidimensional auctions.

The general setting to be analysed here, is a multidimensional procurement auction in which a government or another public authority is the buyer of a good or service and firms or other organizations are potential sellers, who all submit a combination of a price and a quality. The winner of the auction is then determined with the help of a scoring rule. This is a function that gives a value (the score) to all submitted bids, based on all dimensions of the bid. The winning bid will then be the bid obtaining the highest score. After the winner is determined, a contract is made between the buyer and the winning seller in which all important factors are specified.

Three types of multidimensional auctions are considered in this paper. The most natural one is the so-called first-score auction, in which the price-quality combination in the contract will be the same as was specified in the winning bid. An alternative design is the second-score auction, in which the winning bidder has to deliver the service of some quality and for a certain

price, such that the resulting price-quality combination would have obtained the same score as the highest rejected score (so the second highest score). Another type is the second-preferred-offer auction, in which exactly the same price-quality combination is specified in the contract, as the combination obtaining the highest score among the rejected bids.

The main objective of this study is to determine what the maximum possible total welfare resulting from a multidimensional procurement auction can be in certain settings, to derive equilibria for first-score, second-score and second-preferred-offer auctions and to see under which conditions the maximum possible total welfare is obtained in these equilibria. Some of the results which will be obtained are that there exist ex-post equilibria as well as a dominant strategy equilibrium in second-score auctions, while there exist only ex-post equilibria in first-score and second-preferred-offer auctions and that the maximum possible total welfare is obtained in any ex-post equilibrium of a first-score auction in which the scoring rule equals the buyer's true utility function.

5.1 Introduction

"There is scarcely anything in the world that some man cannot make a little worse, and sell a little more cheaply. The person who buys on price alone is this man's lawful prey." -Unknown author[2]

An auction is a mechanism to buy (sell) a product or service, where potential sellers (buyers) have to submit bids, which fully determine the winner of the auction and the price he or she has to pay (or receives). It is a mechanism that was already used long ago and which is known all over the world. All kind of products and services can be auctioned. Famous examples of auctions are the art-auctions in auction houses like Sotheby's and Christie's, which are the two largest auction houses at this moment. However, auctions do not always need to take place in such an auction house, but can actually take place anywhere. A virtual place to auction all kind of items, which is rapidly growing during recent years, is Ebay. Here, anyone is free to submit bids, so everybody can participate in such an auction.

There are a lot of different rules that can be applied in a certain auction. For example, the rules describing when and how often bidders may submit bids, when the auction ends, how the winner will be determined and which amount every participant has to pay, can be totally different between one auction and another. A well-known auction type is the ascending bid auction, in which bidders publicly submit bids and other bidders can overbid by placing a higher bid. The auction ends if nobody overbids anymore and the winner is the bidder who placed the highest bid and pays the amount of his bid. This is also called an English auction and forms an example of a second-price auction. Another type is the Dutch auction, in which a public decreasing price clock is used. Bidders can choose to buy at the price showed by the clock at a certain moment. If this happens, the auction ends and the winner is simply the bidder who agreed to buy at the price and this price is the amount he has to pay. This is an example of a first-price auction. Other commonly used auction types are the first-price sealed bid and second-price sealed bid auctions, in which all bidders can submit one sealed bid and the winner is the bidder with the highest bid, who has to pay his own bid or the second highest bid respectively. A less used, but still interesting type is the all-pay auction, in which only the highest bidder wins, but all participants have to pay their submitted bids.

[2]This quote is often attributed to the work of John Ruskin, an English author and poet of the 19th century. However, there is no evidence that this can be found in any work of Ruskin. Therefore, the origin of this quote remains unknown so far.

All these auction types have in common that bids involve only a certain price for which the bidder wants to buy the product or deliver the service. Although stating a price is enough in some situations, it is not always the only interesting aspect. Auctions in which not only the price, but also other factors are important, are known as multidimensional auctions. The general setting to be analysed here, is a multidimensional procurement auction in which a government or another public authority is the buyer of a good or service and firms or other organizations are potential sellers, who all submit a combination of price $p \in \mathbb{R}_+$ and quality $q \in \mathbb{R}_+$, forming their (sealed) bid $(p, q) \in \mathbb{R}^2_+$ (in what follows, the terms 'firm(s)', 'bidder(s)' and 'seller(s)' will be used next to each other). The winner of the auction is then determined based on the combination of price and quality. Such a procurement auction can be seen as a reversed auction, because the roles of a buyer and a seller are reversed from what is usually the case in an auction (one seller and multiple competing potential buyers). Actually any good or service can be procured. Examples can be simple stationery like pencils and pens, complex building contracts for highways and power plants and anything in between. In the example of the building contract for a highway, quality can be thought of as time to completion and the specific design of the highway. After the winner is determined, a contract is made between the buyer and the winning seller in which all important factors are specified. It is usually assumed that the procured goods and services are indivisible, meaning that there can be only one winning seller per item. Unless stated differently, it is assumed that there is only one good or service procured at a time, so the considered auctions are single-item auctions.

Buyers can have different objectives, for example to maximize total welfare or to maximize own utility. This latter objective is most common. Since a buyer with such goal wants to pay as few as possible (price decreases his utility), but on the same time wants to get the service of a quality as high as possible (quality increases his utility), it is obvious that he prefers a bid (p_1, q_1) over a bid (p_2, q_2), where $p_1 < p_2$ and $q_1 > q_2$. However, more complicated situations can of course also occur. For example, it is not obvious what a buyer would prefer if $p_1 < p_2$ and $q_1 < q_2$. To determine the winning bid in such a case, buyers often use scoring rules. A scoring rule is a function that gives a value (the score) to all submitted bids, based on all dimensions of the bid. So, in the described two-dimensional situation, the scoring rule can be seen as a function $S : \mathbb{R}^2_+ \to \mathbb{R} : (p, q) \to S(p, q)$. The winning bid will then be the bid obtaining the highest score. Auctions in which a scoring rule is used are called scoring auctions. It is commonly assumed that the buyer designs the scoring rule and that this rule is known to all participating sellers before bids are submitted.

In fact, any function that is decreasing in price and increasing in quality, can be used as a scoring rule. At first sight, it sounds plausible that the buyer always chooses the scoring rule to be equal to his own utility function. Namely, this would imply that the winning bid (which is the bid obtaining the highest score) is always the bid which gives the most utility to the buyer. However, there may be several reasons for choosing a scoring rule unequal to the buyer's utility function. The buyer may for example just not know his own utility function. Another reason could be that the utility function is too complex (simple scoring rules such as linear ones are more likely to be well understood in the market and so are likely to perform better).

Sellers have almost always the objective to maximize profit. So, sellers, in contrast to buyers, want to deliver the service for a high price and low costs. Low costs are related to a low quality level, since delivering the same service of a higher quality, usually leads to higher costs. However, submitting bids with a high price and a low quality level, will likely result in losing the auction, which gives 0 profit. It can even lead to a negative profit, a loss, if there are some costs for submitting a bid. Although it is reasonable to assume that all sellers bear higher costs if they have to produce at a higher quality level, their cost functions $c(q)$ are likely to be different. This implies that the same price-quality combination can give different profits to different sellers. Firms therefore have private values, which are given by the price they receive minus the costs for delivering the agreed quality. The firm with the lowest cost function is referred to as the most efficient firm. An outcome of an auction is said to be efficient if and only if the most efficient firm wins. Sometimes cost functions are publicly known, which implies that all firms as well as the buyer know which firm is the most efficient. Other times these form private information for firms. It can also be the case that firms do not know their own cost function. Especially if a complex good or service has to be delivered, firms can probably only estimate costs before the project starts and will first learn about the true costs once the project is implemented or even afterwards.

Like is the case for one-dimensional auctions in which bids only include a price, there also exist different auction types for multidimensional auctions. The most natural one is the so-called first-score auction, in which the price-quality combination in the contract will be the same as was specified in the winning bid. An alternative design is the second-score auction, in which the winning bidder has to deliver the service of some quality and for a certain price, such that the resulting price-quality combination would have obtained the same score as the highest rejected score (so the second highest score). Another type is the second-preferred-offer auction, in which exactly the same price-quality combination is specified in the contract, as the combination obtaining the highest score among the rejected bids. Clearly, the second-score and second-preferred-offer auctions can be seen as the multi-dimensional versions of second-price auctions.

The main objective of this study is to determine what the maximum possible total welfare resulting from a multidimensional procurement auction can be in certain settings, to derive equilibria for first-score, second-score and second-preferred-offer auctions and to see under which conditions the maximum possible total welfare is obtained in these equilibria.

In section 2, the performance of multidimensional auctions is compared to the performance of one-dimensional auctions and negotiations. In section 3, the maximum possible total welfare which can be derived from a multidimensional auction is calculated for some settings. Next, in section 4, equilibria will be derived for first-score auctions and conditions are obtained under which the earlier derived maximum possible total welfare is achieved in these equilibria. Distinction will be made between cases in which the scoring rule equals the utility function of the buyer and cases in which these are different from each other. Section 5 summarises, concludes and discusses some possible topics for future research.[3]

5.2 Comparing award mechanisms

Before going into further detail of multidimensional auctions, it may be interesting to know why they are actually used and what their advantages and disadvantages are compared to alternative award mechanisms. In procurement markets, the most common alternatives are one-dimensional auctions and negotiations. The discussion about which award mechanism should be used under what circumstances is important, since procurement represents a significant part of economic activity, so problems and inefficiencies caused by the wrong choice of award mechanism can have a large influence on economies. There is always a trade-off between theoretical benefits and drawbacks of the different mechanisms. While such benefits may be stronger than the drawbacks in a specific setting, in another setting it can very well be the case that the drawbacks play a more important role. So, which mechanism performs the best (in terms of efficiency, cost savings by the government or total welfare) in a certain situation seems to depend heavily on the specific characteristics of the considered procurement market.

[3] In the full version of this paper, all of these subjects are discussed in more detail. Moreover, the original work includes an extensive literature review and similar to the analysis of first-score auctions, analyses of second-score and second-preferred-offer auctions are presented.

There exists a large literature about the pros and cons of award mechanisms. An important argumentation in favour of auctions is given by [Bulow and Klemperer(1996)], who show that the increased competition by one extra bidder in an auction more than offsets the loss of any negotiation power, in terms of cost savings for the buyer. They therefore conclude that auctions should always be used, as long as there is at least one more firm likely to participate if an auction is used instead of a negotiation. Other arguments favouring auctions are that the most efficient firm is more likely to win in a competitive environment, that they give equal opportunities to firms and that they increase transparency in the market.

A main argument against the use of auctions is given by [Goldberg(1977)], who argues that communication and coordination are important to find an eligible seller and to be able to make suitable contracts before the project is implemented, especially if the procured project is complex. Such communication and coordination is of course easier if negotiations are used, since the buyer can then address every potential seller in turn to bargain about a possible contract and during this bargaining, communication and coordination will take place automatically. Another argument against auctions is that they create more opportunities for firms to collude, since firms can make arrangements about the bids they are going to submit.

An argument specifically against one-dimensional auctions is that renegotiations are more likely to be needed if quality dimensions are not considered when determining the winning firm. Namely, this firm will then deliver a low quality (in fact, the lowest possible quality), since this leads to lower costs, for the specified price. To prevent this, the buyer has to ask for a renegotiation of the contract. The firm, which is then already be chosen to deliver the good or service, can then ask for a much higher price in response to a higher quality level. Moreover, the renegotiation itself takes valuable time.

[Manelli and Vincent(1995)] also argue that one-dimensional auctions perform bad if quality is an important and variable factor, since only products of the lowest quality would then be delivered. In their theoretical analysis, award mechanisms are modelled as infinite dimensional programs, which are solved by solving the dual programs. The optimality for society as a whole is compared between outcomes of second-price auctions with reserve prices and mechanisms in which the buyer sequentially offers a take-it-or-leave-it price to firms until one firm accepts the offer (this can be seen as a form of negotiation). Several necessary and sufficient conditions are obtained from which the result follows that the sequential offer mechanism leads in more cases to the socially optimal outcome than the considered auction.

At first sight, multidimensional auctions seem to form a perfect solution for all problems related to one-dimensional auctions and negotiations. Namely, general benefits of auctions are combined with the benefit of addressing quality in the bidding process, which is a major advantage of negotiations. However, although these benefits are all valid for multidimensional auctions, they still suffer from some drawbacks of the discussed alternatives and also have their own problems.

A frequently used argument against multidimensional auctions is that they are often too complicated to implement. Buyers may for example have difficulties with designing appropriate scoring rules and sellers may have problems with understanding how the auction exactly works. Moreover, like is the case for negotiations, corruption and favouritism of the buyer can be a significant problem. A theoretical analysis of multidimensional auctions in which possible corruption is modelled, is given by [Burguet and Che(2004)].

[Estache et al.(2009)] argue that despite quality is considered in the bidding process, if there are only a few potential sellers participating in the auction, renegotiations are even more likely to take place in multidimensional auctions than in one-dimensional auctions. They argue that if the buyer wants to achieve several goals (for example a low price as well as a high quality), it is possible that every goal is achieved partly, which increases the need for renegotiations. If the buyer would instead focus on one specific goal (for example a low price), it would be more likely that this goal is fully achieved. From the empirical analysis, in which data is used from complex road and railroad projects in 11 Latin American and Caribbean countries, it is concluded that it indeed seems to be the case that, if there are only a few firms submitting bids, renegotiations are more often needed for contracts awarded by multidimensional auctions. This is in line with the findings of [Cabizza and De Fraja(1998)], who consider auctions for television franchises in which quality is measured by the quality of broadcasted programmes. They find that in case of limited competition, multidimensional auctions lead relatively often to inefficient outcomes and this in turn is positively related to the need for renegotiations.

One has to realise that the provided arguments are mainly of theoretical importance. In practice, it can very well be the case that multidimensional auctions perform better than their alternatives. It should therefore not be surprising that multidimensional auctions are indeed frequently used in all kinds of situations. This confirms that it is important to understand the implications of such auctions and justifies the need for a more detailled analysis.

5.3 The maximum possible total welfare

Consider a two-dimensional procurement auction in which there are N potential sellers of the service to be procured. The two dimensions are price p and quality q which both have to be specified in a (sealed) bid $(p, q) \in \mathbb{R}^2_+$. As a start, let us have a look at the optimal result from the perspective of society as a whole. This optimal result would be an outcome leading to the maximum possible total welfare created by the procurement auction. To this purpose, it is not necessary yet to introduce scoring rules or other characteristics of the auction. Assume that the utility function of the buyer has the form

$$U(p, q) = V(q) - p, \tag{5.1}$$

where $V(q)$ represents how the buyer values quality q and can be any increasing function. A quality of 0 will give 0 value to the buyer, so $V(0) = 0$. Note that this utility function is quasi-linear. Furthermore assume that the profit function of seller $i \in \{1, 2, ..., N\}$, which is actually his utility function, is given by

$$\Pi_i(p, q) = \begin{cases} p - c_i(q) & \text{if } i \text{ is the winning seller,} \\ 0 & \text{if } i \text{ is not the winning seller,} \end{cases} \tag{5.2}$$

where $c_i(q)$ is the strictly increasing cost function of firm i.[4] There will be no costs for delivering a quality of 0, so $c_i(0) = 0$. Note that this profit function is quasi-linear as well. By combining the utility function with the profit function, it is easy to see that total welfare (TW), conditional on firm i winning the auction, is given by

$$\begin{aligned} TW &= (V(q) - p) + (p - c_i(q)) \\ &= V(q) - c_i(q). \end{aligned}$$

Note that this only depends on quality, not on price. This result stems from the fact that both the utility function and the profit function are linear in price (which makes them quasi-linear). Namely, this means that the price has only distributional effects. Although distribution of wealth is usually an important factor in determining what is best for society, this does not influence the optimal outcome here, since the optimal outcome is now just defined as the outcome maximizing total welfare, independent of how equally this welfare is distributed. Moreover, note that in order to

[4] In the remainder, the profit function will usually only be given for the case of winning. However, it will always remain true that the profit is 0 for all losing firms (since no costs for submitting bids or other entry-costs are modelled).

achieve the maximum total welfare, the firm with the lowest cost function, so the most efficient firm, should be the winner. These observations are related to the famous Value Maximization Principle, which was first showed in [Milgrom and Roberts(1992)]. This principle states that if utilities are quasi-linear, an outcome maximizes total utility if and only if the outcome is Pareto optimal if and only if the outcome is efficient.

By specifying the functions $V(q)$ and $c_i(q)$, the optimal quality level can be derived. Before doing so, however, it is important to realise two things. First, the derived quality is optimal for society as a whole, not for the firm who has to offer a quality in its bid. The firm namely wants to maximize its profit, not total welfare (except of course for cases in which these goals require the same strategy of the firm). It is therefore unlikely that the firm will indeed choose this optimal quality level, so the results will mainly have a theoretical meaning. Second, the derived quality is optimal if this is the quality specified in the contract between the buyer and the winning firm. This does not have to mean that this should also be the proposed quality in the winning bid, since, as mentioned before, the price-quality combination in this winning bid does not always have to be the same as the price-quality combination in the final contract (remember the second-score and second-preferred-offer designs).

To start with an easy example, let $V(q) = q$ and $c_i(q) = iq^2$ for all $i \in \{1, 2, ..., N\}$. Observe that in this case, firm 1 has the lowest cost function, so the allocation will be efficient if and only if firm 1 wins the auction. The cost function is convex and implies that there are increasing marginal costs. The maximization problem becomes

$$\max_q (q - iq^2) \text{ subject to } q \geq 0.$$

Taking the first derivative leads to

$$\begin{aligned} 1 - 2iq &= 0 \\ &\Leftrightarrow \\ q &= 1/(2i) \end{aligned}$$

and the second derivative

$$-2i < 0 \; \forall \; i = 1, 2, ..., N$$

shows that it is indeed a maximum. For firm 1 we get $q = 1/2$ and so a total welfare of

$$TW_1 = 1/2 - (1/2)^2$$

$$= \quad 1/4$$

if 1 wins. For firm 2 we get $q = 1/4$ and so a total welfare of

$$\begin{aligned} TW_2 &= 1/4 - 2 * (1/4)^2 \\ &= 1/8 \\ &< 1/4 \\ &= TW_1 \end{aligned}$$

if 2 wins. For firm 3 we get $q = 1/6$ and so a total welfare of

$$\begin{aligned} TW_3 &= 1/6 - 3 * (1/6)^2 \\ &= 1/12 \\ &< 1/4 \\ &= TW_1 \end{aligned}$$

if 3 wins and so on. Indeed the result is obtained that in order to achieve maximum total welfare, the most efficient firm should win.

Now consider a more general example, by letting $V(q) = aq$ and $c_i(q) = k_i q^2$, where $a, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. An efficient firm is now a firm m with $k_m \leq k_n$ for all firms $n \neq m$. Denote k_m by $\underline{k}_i$. The corresponding maximization problem is

$$\max_q (aq - k_i q^2) \text{ subject to } q \geq 0.$$

The first derivative yields

$$\begin{aligned} a - 2k_i q &= 0 \\ &\Leftrightarrow \\ q &= a/(2k_i), \end{aligned}$$

while the second derivative

$$-2k_i < 0 \; \forall \; i = 1, 2, ..., N$$

confirms that this is indeed a maximum. For firm m we get $q = a/(2\underline{k}_i)$ and a total welfare of

$$\begin{aligned} TW_m &= a * (a/(2\underline{k}_i)) - \underline{k_i} * (a/(2\underline{k}_i))^2 \\ &= (1/4)a^2/\underline{k}_i \end{aligned}$$

if m wins. For any other firm n we get $q = a/(2k_n)$ and a total welfare of

$$TW_n = a * (a/(2k_n)) - k_n * (a/(2k_n))^2$$

$$
\begin{aligned}
&= (1/4)a^2/k_n \\
&\leq (1/4)a^2/\underline{k}_i \\
&= TW_m
\end{aligned}
$$

if n wins. Again the result is obtained that an efficient firm should win for total welfare to be at the highest possible level. Next, let $V(q) = bq^2$ and $c_i(q) = k_i q^3$, where $b, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. The maximization problem is now

$$\max_q (bq^2 - k_i q^3) \text{ subject to } q \geq 0.$$

The first derivative implies

$$
\begin{aligned}
2bq - 3k_i q^2 &= 0 \\
&\Leftrightarrow \\
q &= (2b)/(3k_i) \text{ (or } q = 0),
\end{aligned}
$$

while the second derivative at $q = (2b)/(3k_i)$ is

$$2b - 6k_i(2b)/(3k_i) = -2b < 0 \;\forall\; i = 1, 2, ..., N,$$

which ensures a maximum. If an efficient firm m wins, $q = (2b)/(3\underline{k}_i)$ and total welfare equals

$$
\begin{aligned}
TW_m &= b * ((2b)/(3\underline{k}_i))^2 - \underline{k_i} * ((2b)/(3\underline{k}_i))^3 \\
&= (4/9)b^3/\underline{k}_i^2 - (8/27)b^3/\underline{k}_i^2 \\
&= (4/27)b^3/\underline{k}_i^2.
\end{aligned}
$$

If another firm n wins, $q = (2b)/(3k_n)$ and total welfare will be equal to

$$
\begin{aligned}
TW_n &= b * ((2b)/(3k_n))^2 - k_n * ((2b)/(3k_n))^3 \\
&= (4/9)b^3/k_n^2 - (8/27)b^3/k_n^2 \\
&= (4/27)b^3/k_n^2 \\
&\leq (4/27)b^3/\underline{k}_i^2 \\
&= TW_m.
\end{aligned}
$$

Like one could expect, also here total welfare is at the highest possible level only if an efficient firm wins.

To look at one more example, let $V(q) = e\sqrt{q}$ and $c_i(q) = k_i\sqrt{q}$, where $e, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. In contrast to the examples before, the cost function is now concave and implies that there are decreasing marginal costs. Another important difference with foregoing examples is that now the functions $V(q)$ and $c_i(q)$ 'depend both in the same way on q'. In other words, the parts in these functions which include q have the same form, namely both take the square root of this argument. This means that we can simplify total welfare to

$$\begin{aligned} TW &= e\sqrt{q} - k_i\sqrt{q} \\ &= (e - k_i)\sqrt{q}. \end{aligned}$$

It is helpful to split this problem into 3 cases. First, if $e < k_i$, total welfare decreases if quality increases so it is best if quality is as low as possible. This would imply $q = 0$, but since this may be difficult to interpret, suppose for this moment that there is a minimum quality $\underline{q}$ required. If $e < \underline{k}_i$, it is straightforward to see that the efficient firm m should win and quality should be chosen at $\underline{q}$ to maximize total welfare, which is then $(e - \underline{k}_i)\sqrt{\underline{q}}$. Second, if $e = k_i$, total welfare is independent of quality so quality can be chosen at any level. In the situation that $e = \underline{k}_i$ it is obvious that the efficient firm m should win again and total welfare will be 0, independent of the delivered quality. Third, if $e > k_i$, total welfare increases if quality increases so it is best if quality is as high as possible. Theoretically quality should approach ∞, but since this is difficult to interpret as well, assume at this point that there is a maximum quality $\bar{q}$. Also in case that $e > \underline{k}_i$ it is easy to see that total welfare is maximized by letting the efficient firm m win and setting quality equal to $\bar{q}$. This total welfare will then be $(e - \underline{k}_i)\sqrt{\bar{q}}$.

Of course, similar calculations can be performed for other combinations of functions $V(q)$ and cost functions $c_i(q)$. The results for several such combinations are presented in the complete version of this paper. It goes without saying that the maximum total welfare can always only be achieved if the winner is among the most efficient firms. For some combinations, like $V(q) = bq^2$ with $c_i(q) = k_i q$, there turns out to be an interior minimum, but no interior maximum and total welfare is then maximized by choosing $q = \bar{q}$.

After calculating the quality that should be chosen to maximize total welfare and the corresponding amount of total welfare for several combinations of $V(q)$ and cost functions $c_i(q)$, it is known what the best possible outcome for society is. However, this theoretical outcome is not at all guaranteed to be the real outcome of some auction. From now on, the main objective is to see what will actually happen in (equilibria of) certain auctions and under which conditions the derived maximum possible total welfare is obtained.

5.4 First-score auctions

In this section, first-score auctions are considered, so auctions in which the winning price-quality bid coincides with the price-quality combination in the final contract. Let there be a publicly known scoring rule $S : \mathbb{R}^2_+ \to \mathbb{R} : (p, q) \to S(p, q)$, giving each submitted bid a score. The bid obtaining the highest score is the winning bid. Assume that in case of a draw, the most efficient firm among the firms involved in this draw, will be awarded the contract.[5] Another important assumption is that the cost function of every firm is publicly known. This means in particular that every firm knows which of them is the most efficient one. Such an efficient firm will again be denoted by firm m, while all other firms are reported as a firm n and also all other functions and characteristics, for example the general form of utility functions and profit functions (see equation (5.1) and (5.2)), are unchanged from the foregoing discussion.

5.4.1 Equilibria in first-score auctions

To begin with, equilibria in a first-score auction have to be specified. It will be argued that there exist (infinitely many) ex-post equilibria. The concept of such an ex-post equilibrium in an auction setting was first mentioned in [Crémer and McLean(1985)] and has become a widely used topic in auction theory. The notion of an ex-post equilibrium is weaker than a dominant strategy equilibrium, but stronger than a Bayesian Nash equilibrium. The bids of all sellers together form an ex-post equilibrium if the bid from every seller $i \in \{1, 2, ..., N\}$ is a best reply, given the bids from all other sellers. It says that no seller i will have regret after observing the other bids, since if all other bids were already known to i at the time at which he had to submit his bid, he could not have done better by choosing any other bid. A strategy is a dominant strategy if it is a best reply, no matter what other bidders bid. So this means that the bidder could not have done better, independent of other bids.

An example of an ex-post equilibrium in the setting at hand, is the situation in which every inefficient firm n maximizes the obtained score (thereby maximizing the probability of winning) such that it does not make a loss (if it would make a loss in case of winning, it would prefer to lose the auction) and a firm m maximizes its profit such that it reaches the same

[5] In case the reader does not find this assumption plausible, realise that all variables and functions used in the model are continuous. This implies that the most efficient firm could change one dimension of its bid by a very small $\epsilon > 0$, thereby obtaining a strictly higher score. Then the most efficient firm is sure of winning, without changing other aspects such as profit significantly, even if this assumption does not hold.

score as the highest score among the other bids. So the most efficient firm wins and makes a nonnegative profit. The constraint for firm n will boil down to making a profit of exactly 0, since if it would submit a bid such that it would obtain a strictly positive profit in case of winning, it could have obtained a higher score without making a loss and this is then not in line with the above described strategy anymore.[6]

After observing all bids, a firm n knows that it could not have done better than losing and thereby getting a profit of 0. Namely, in order to win, a bid was required that would have obtained a strictly higher score. This in turn, would have caused a strictly negative profit. Also firm m realises that it could not have done better, since submitting a bid obtaining a lower score would have led to losing the auction (and so a profit of 0), submitting another bid that would have reached the same score could not have led to a higher profit by construction and if submitting a bid that would have led to a higher score would have led to a higher profit, it could also have earned a higher profit by obtaining the same score as it obtains now, which is in contradiction with maximizing profit conditional on reaching the highest score among the other bids. This means that the described situation indeed forms an ex-post equilibrium. This particular ex-post equilibrium will from now on be denoted by FSC.

The outcome is visualised in figure 5.1. The straight lines represent scores, so points on the same straight line are price-quality combinations leading to the same score. The concave curves represent profits, so points on the same concave curve are price-quality combinations leading to the same profit. These forms are chosen to simplify the figure. However, one has to keep in mind that the basic idea of the analysis will be the same if the score-lines or the profit-curves have another form. To prevent the figure from becoming too confusing, only the behaviour of the most efficient firm and the second most efficient firm is demonstrated (note that this is sufficient to see what happens in the FSC-equilibrium). This second most efficient firm will from now on always be denoted by firm $\tilde{n}$. So firm $\tilde{n}$ can be seen as the most efficient firm among firms n.

The line $\bar{S}_m$ is the highest score which can still lead to a nonnegative profit for firm m and the line $\bar{S}_{\tilde{n}}$ is the highest score which can still lead to a nonnegative profit for firm $\tilde{n}$. The highest score for firm $\tilde{n}$ conditional on making 0 profit is obtained by bidding $(p_{\tilde{n}}, q_{\tilde{n}})$. Firm m maximizes profit conditional on reaching the highest score among the other bids (by construction this is the score obtained by firm $\tilde{n}$) by bidding (p_m, q_m). By

[6]To keep this profitable for the firm, one could also say that firm n has to maximize the score, such that the profit in case of winning is equal to a very small $\epsilon > 0$. This does not change the basic idea of the analysis.

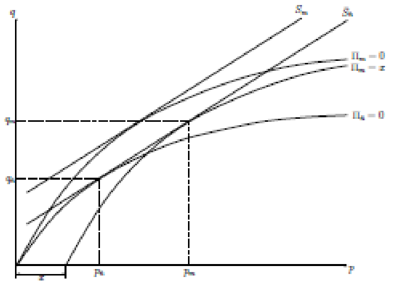

Figure 5.1: FSC-equilibrium.

doing so, (p_m, q_m) will be the winning bid and the profit for firm m becomes x, because the winning combination is on the curve $\Pi_m = x$, which is obtained by shifting the curve $\Pi_m = 0, x$ to the right. In the remainder of the analysis of the first-score auction, the FSC-equilibrium will be considered in more depth.

Before going on with this analysis, however, let us think about what the other ex-post equilibria actually are. Imagine that a firm n wins. Then it follows directly that either this firm makes a loss and could have done better by losing or firm m could have done better by obtaining the currently winning score. Moreover, if the winning firm does not maximize its profit conditional on obtaining the score with which it now wins, obviously it could have done better. Furthermore, if the winning bid obtains a higher score than the highest score which could still lead to a nonnegative profit for the most efficient firm, the winner, independent of which firm this is, makes a loss and could therefore have done better by submitting a bid that would have resulted in losing. A similar idea can be applied if the winning bid obtains a lower score than the highest score which could still lead to a nonnegative profit for firm $\tilde{n}$. Namely, if firm m or $\tilde{n}$ would win, firm $\tilde{n}$ or respectively m could have done better by obtaining a score between the currently winning score and the mentioned highest score which could still lead to a nonnegative profit for firm $\tilde{n}$; if another firm would win, both m and $\tilde{n}$ could have done better. Also an outcome in which the winning bid obtains a higher score than the second best bid cannot be an ex-post equilibrium, since in this case the winning firm could have done better by obtaining a score in between its current score and the score of the second best bid, thereby earning a profit which is higher than the maximum possible profit when obtaining the current score.

Taking these results together one can see that in order to form an ex-post equilibrium, it is necessary that the most efficient firm wins, that it maximizes profit conditional on obtaining the currently winning score, that the winning bid obtains the same score as the second best bid and that the winning score is lower than or equal to the highest score which could still lead to a nonnegative profit for firm m and on the same time higher than or equal to the highest score which could still lead to a nonnegative profit for firm $\tilde{n}$. In fact, these necessary conditions are together also sufficient, which is formally stated in proposition 1.

Proposition 1. *Consider a first-score auction with a tie-breaking rule such that the most efficient firm involved in the draw will win. Let firm m, n and $\tilde{n}$ be defined as before. Let there be bids such that all of the following conditions are satisfied:*

1. *Firm m wins the auction;*
2. *The winning firm maximizes profit conditional on obtaining the currently winning score;*
3. *The winning bid obtains the same score as the second best bid;*
4. *The winning score is lower than or equal to the highest score which could still lead to a nonnegative profit for firm m;*
5. *The winning score is higher than or equal to the highest score which could still lead to a nonnegative profit for firm $\tilde{n}$.*

Then these bids form an ex-post equilibrium.

Proof. Consider an arbitrary firm n. This firm now makes 0 profit, since it loses (condition 1). Possibly, it could have been better off by winning, but by conditions 1, 5 and the described tie-breaking rule, this would require obtaining a score strictly higher than the highest score which could still lead to a nonnegative profit for firm $\tilde{n}$. Instead of being better off, firm n would now even be strictly worse off. Now consider firm m, which makes a nonnegative profit by conditions 2 and 4. If its bid obtained a lower score than the currently obtained score, it would lose the auction by condition 3 and would therefore make 0 profit, which cannot be strictly better than its current profit. If it submitted another bid obtaining the same score, it could not have been better off by condition 2. Finally, think about the situation in which it would have reached a higher score. If it now would be better off, it could also have been better off by submitting another bid obtaining the currently winning score. This is in contradiction with condition 2. Therefore, the bid of each firm is a best response, given the other firm's bids, which means that the bids form an ex-post equilibrium. □

Figure 5.2 provides a visual representation of a general ex-post equilibrium. Again, to prevent the figure from becoming too confusing, only the behaviour of two firms, the winning firm m and a firm n (for example, but not necessary, firm $\tilde{n}$) which obtains the second highest score, is demonstrated. It is assumed that all other firms obtain a lower score than the score of $\hat{S}$ (represented by the line in the middle), which makes the behaviour of these other firms unimportant in the current analysis. It is not difficult to see that also in this figure, all necessary conditions are indeed satisfied.

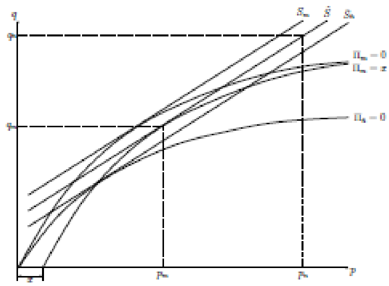

Figure 5.2: General ex-post equilibrium in first-score auction.

Note that in the FSC-outcome, no firm would make a loss (strictly negative profit) if its bid turned out to be the winning bid. It is of course reasonable to assume that also in practice no firm would like to take the risk of making a loss if it cannot make a positive profit anyway, but instead would prefer to lose the auction, resulting in 0 profit for sure. However, in some other ex-post equilibria, firms do submit bids which would lead to a loss if it turned out to be the winning bid, although this is of course not very realistic to happen. To be more precise, ex-post equilibria in which this does not occur are only the FSC-equilibrium and its variants in which firm m and $\tilde{n}$ do exactly the same as in the FSC-equilibrium and other firms submit anything that would not lead to a loss in case of winning. Observe that the FSC-equilibrium and all these variants will have the same outcome in terms of total welfare and price-quality combination in the final contract.

Note furthermore that there does not exist a dominant strategy for the firms. This follows directly from the fact that the winning firm would like to reach the second highest score, but not a strictly higher score. Since this second highest score is not fixed, the best reply of the winning firm is also subject to change. For example, let there be a firm which can reach a score of 10 without making a loss. If the highest obtained score among the other bids is 8, the firm would like to win with reaching a score of 8 as well. If the highest obtained score among the other bids is 6, the firm would like to win again, but now with a score of 6 instead of 8. If it would still have obtained a score of 8, it could have done better in this last situation. From this simple example it becomes already clear that there is no strategy forming a best reply independent of the other bids, so indeed there does not exist a dominant strategy.

5.4.2 Obtaining maximum possible total welfare in first-score auctions if the scoring rule equals buyer's utility function

After having discussed in words what will happen in the FSC-equilibrium (and other ex-post equilibria), it is now time for the quantitative analysis of the FSC-equilibrium by looking at several numerical examples (only two examples are presented here). Assume for this moment that the scoring rule corresponds to the true utility function of the buyer, so $S(p,q) = U(p,q)$.

As a first example, let $V(q) = aq$ and $c_i(q) = k_i q^2$, where $a, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. The scoring rule becomes

$$S(p,q) = U(p,q) = aq - p$$

and the profit function of any firm i can now be written as

$$\Pi_i(p,q) = p - k_i q^2.$$

An efficient firm m now has $k_m \leq k_n$ for all firms $n \neq m$ and like was done before, denote k_m by $\underline{k}_i$. Besides, let the most efficient firm among all other firms, firm $\tilde{n}$, have $k_{\tilde{n}} = \underline{k}_n$. This notation will from now on always be used. Firm $\tilde{n}$ now has to solve

$$\max_{p,q}(aq - p) \text{ subject to } p - \underline{k}_n q^2 = 0 \text{ and } p, q \geq 0.$$

Substitution yields

$$\max_{q}(aq - \underline{k}_n q^2) \text{ subject to } q \geq 0,$$

the first derivative gives

$$a - 2\underline{k}_n q \quad = \quad 0$$

$$\begin{array}{rcl} & \Leftrightarrow & \\ q & = & a/(2\underline{k}_n) \end{array}$$

and the second derivative

$$-2\underline{k}_n < 0$$

assures a maximum. Then $p = \underline{k}_n(a/(2\underline{k}_n))^2 = (1/4)a^2/\underline{k}_n$ and $S = a^2/(2\underline{k}_n) - (1/4)a^2/\underline{k}_n = (1/4)a^2/\underline{k}_n$. Firm m now solves

$$\max_{p,q}(p - \underline{k}_i q^2) \text{ subject to } aq - p = (1/4)a^2/\underline{k}_n \text{ and } p, q \geq 0,$$

which becomes

$$\max_{q}((aq - (1/4)a^2/\underline{k}_n) - \underline{k}_i q^2) \text{ subject to } q \geq 0$$

after substitution. Use the first derivative to get

$$\begin{array}{rcl} a - 2\underline{k}_i q & = & 0 \\ & \Leftrightarrow & \\ q & = & a/(2\underline{k}_i) \end{array}$$

and the second derivative

$$-2\underline{k}_i < 0$$

to show that it is a maximum. This leads to $p = (1/2)a^2/\underline{k}_i - (1/4)a^2/\underline{k}_n$. Utility for the buyer equals $U = (1/4)a^2/\underline{k}_n$, profit for firm m is $\Pi_m = (1/4)a^2/\underline{k}_i - (1/4)a^2/\underline{k}_n$ and total welfare the sum of these, so $(1/4)a^2/\underline{k}_i$.

Note that this outcome is the same as the outcome maximizing total welfare, as can be seen in the corresponding example in the section about the maximum possible total welfare. As another example, choose $V(q) = e\sqrt{q}$ and $c_i(q) = k_i\sqrt{q}$, where $e, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. This results in a scoring rule looking like

$$S(p, q) = U(p, q) = e\sqrt{q} - p$$

and profit for firm i can now be calculated by

$$\Pi_i(p, q) = p - k_i\sqrt{q}.$$

Similar to what was assumed in the last example of the total welfare analysis, assume that there is a minimum quality level $\underline{q}$ and a maximum quality level $\bar{q}$. This implies for firm $\tilde{n}$ that it has to solve

$$\max_{p,q}(e\sqrt{q} - p) \text{ subject to } p - \underline{k}_n\sqrt{q} = 0, p \geq 0 \text{ and } q \in [\underline{q}, \bar{q}],$$

which can be simplified to

$$\max_q (e\sqrt{q} - \underline{k}_n\sqrt{q}) \text{ subject to } q \in [\underline{q}, \bar{q}]$$

or equivalently to

$$\max_q ((e - \underline{k}_n)\sqrt{q}) \text{ subject to } q \in [\underline{q}, \bar{q}].$$

As one can see from this last expression, three cases need to be distinguished.

Let $\underline{\text{case 1}}$ be $e < \underline{k}_n$ (which does not say anything about the relation between e and $\underline{k}_i$). Now, firm $\tilde{n}$ will choose $\underline{q}$, giving $p = \underline{k}_n\sqrt{\underline{q}}$ and $S = (e - \underline{k}_n)\sqrt{\underline{q}}$. Firm m therefore now faces

$$\max_{p,q} (p - \underline{k}_i\sqrt{q}) \text{ subject to } e\sqrt{q} - p = (e - \underline{k}_n)\sqrt{\underline{q}}, p \geq 0 \text{ and } q \in [\underline{q}, \bar{q}].$$

This can also be written as

$$\max_q (e - \underline{k}_i)\sqrt{q} - (e - \underline{k}_n)\sqrt{\underline{q}} \text{ subject to } q \in [\underline{q}, \bar{q}].$$

Three subcases have to be considered.

Let $\underline{\text{case } 1\alpha}$ be $e > \underline{k}_i$. Then m will choose $\bar{q}$ and as a consequence $p = e\sqrt{\bar{q}} - (e - \underline{k}_n)\sqrt{\underline{q}}$, $U = (e - \underline{k}_n)\sqrt{\underline{q}}$ and $\Pi_m = (e - \underline{k}_i)\sqrt{\bar{q}} - (e - \underline{k}_n)\sqrt{\underline{q}}$. Together, these results show that total welfare equals $(e - \underline{k}_i)\sqrt{\bar{q}}$.

Let $\underline{\text{case } 1\beta}$ be $e = \underline{k}_i$. Then m will choose any quality level $\hat{q}$ and therefore $p = e\sqrt{\hat{q}} - (e - \underline{k}_n)\sqrt{\underline{q}}$, $U = (e - \underline{k}_n)\sqrt{\underline{q}}$ and $\Pi_m = (e - \underline{k}_i)\sqrt{\hat{q}} - (e - \underline{k}_n)\sqrt{\underline{q}}$. This implies a total welfare of $(e - \underline{k}_i)\sqrt{\hat{q}} = 0$.

Let $\underline{\text{case } 1\gamma}$ be $e < \underline{k}_i$. Then m will choose $\underline{q}$ and as a result $p = e\sqrt{\underline{q}} - (e - \underline{k}_n)\sqrt{\underline{q}} = \underline{k}_n\sqrt{\underline{q}}$, $U = (e - \underline{k}_n)\sqrt{\underline{q}}$ and $\Pi_m = (e - \underline{k}_i)\sqrt{\underline{q}} - (e - \underline{k}_n)\sqrt{\underline{q}} = (\underline{k}_n - \underline{k}_i)\sqrt{\underline{q}}$. Thus total welfare becomes $(e - \underline{k}_i)\sqrt{\underline{q}}$.

Let $\underline{\text{case 2}}$ be $e = \underline{k}_n$ (which implies $e \geq \underline{k}_i$, assuming that the two most efficient firms can be equally efficient). Now, firm $\tilde{n}$ will choose any quality level $\tilde{q}$, resulting in $p = \underline{k}_n\sqrt{\tilde{q}}$ and $S = (e - \underline{k}_n)\sqrt{\tilde{q}}$. Firm m now has to solve

$$\max_{p,q} (p - \underline{k}_i\sqrt{q}) \text{ subject to } e\sqrt{q} - p = (e - \underline{k}_n)\sqrt{\tilde{q}}, p \geq 0 \text{ and } q \in [\underline{q}, \bar{q}].$$

After substitution this becomes

$$\max_q (e - \underline{k}_i)\sqrt{q} - (e - \underline{k}_n)\sqrt{\tilde{q}} \text{ subject to } q \in [\underline{q}, \bar{q}].$$

Two subcases can be distinguished.

Let $\underline{case\ 2\alpha}$ be $e > \underline{k}_i$. Then m will choose $\bar{q}$, so $p = e\sqrt{\bar{q}} - (e - \underline{k}_n)\sqrt{\bar{q}}$, $U = (e - \underline{k}_n)\sqrt{\bar{q}}$ and $\Pi_m = (e - \underline{k}_i)\sqrt{\bar{q}} - (e - \underline{k}_n)\sqrt{\bar{q}}$. Total welfare therefore becomes $(e - \underline{k}_i)\sqrt{\bar{q}}$.

Let $\underline{case\ 2\beta}$ be $e = \underline{k}_i$. Then m will choose any quality $\hat{q}$, implying $p = e\sqrt{\hat{q}} - (e - \underline{k}_n)\sqrt{\bar{q}}$, $U = (e - \underline{k}_n)\sqrt{\bar{q}}$ and $\Pi_m = (e - \underline{k}_i)\sqrt{\hat{q}} - (e - \underline{k}_n)\sqrt{\bar{q}}$. Total welfare now equals $(e - \underline{k}_i)\sqrt{\hat{q}} = 0$.

Let $\underline{case\ 3}$ be $e > \underline{k}_n$ (which implies $e > \underline{k}_i$). Now, firm $\tilde{n}$ will choose $\bar{q}$, leading to $p = \underline{k}_n\sqrt{\bar{q}}$ and $S = (e - \underline{k}_n)\sqrt{\bar{q}}$. For firm m the maximization problem now becomes

$$\max_{p,q}(p - \underline{k}_i\sqrt{q}) \text{ subject to } e\sqrt{q} - p = (e - \underline{k}_n)\sqrt{\bar{q}}, p \geq 0 \text{ and } q \in [\underline{q}, \bar{q}],$$

which can also be expressed as

$$\max_{q}(e - \underline{k}_i)\sqrt{q} - (e - \underline{k}_n)\sqrt{\bar{q}} \text{ subject to } q \in [\underline{q}, \bar{q}].$$

Since $e > \underline{k}_i$, firm m chooses $\bar{q}$. So, $p = e\sqrt{\bar{q}} - (e - \underline{k}_n)\sqrt{\bar{q}} = \underline{k}_n\sqrt{\bar{q}}$, $U = (e - \underline{k}_n)\sqrt{\bar{q}}$ and $\Pi_m = (\underline{k}_n - \underline{k}_i)\sqrt{\bar{q}}$. So total welfare becomes $(e - \underline{k}_i)\sqrt{\bar{q}}$.

Note again that just like in the previous example, the outcome for every subcase obtained here is equal to the outcome which maximizes total welfare, as can be seen in the corresponding example in the section about the maximum possible total welfare.

Of course it is no coincidence that maximum total welfare is achieved in both examples. As one could expect, this will be the case for any combination of functions $V(q)$ and $c_i(q)$. This is exactly the message of proposition 2. See the complete version of this paper for the proof of this proposition.

Proposition 2. *Consider a first-score auction in which the scoring rule equals the buyer's true utility function. Let this utility function be of the general form in equation* (5.1)*. Let the profit function of a firm be of the general form in equation* (5.2)*. Then the maximum possible total welfare is obtained in the FSC-equilibrium, independent of the specific form of the functions* $V(q)$ *and* $c_i(q)$*.*

From proposition 2 it follows that the maximum possible total welfare is always obtained in the FSC-equilibrium, if the scoring rule equals the utility function of the buyer. Proposition 3 generalizes proposition 2 and states that the maximum possible total welfare is obtained in every ex-post equilibrium. For the proof of this proposition, the interested reader is once more referred to the complete version of this paper.

Proposition 3. *Consider a first-score auction in which the scoring rule equals the buyer's true utility function. Let this utility function be of the general form in equation* (5.1)*. Let the profit function of a firm be of the general form in equation* (5.2)*. Then the maximum possible total welfare is obtained in any ex-post equilibrium.*

Although the same quality will be implemented in each ex-post equilibrium (given some functions $V(q)$ and $c_i(q)$), the obtained scores by winning bids can be different. Note that the same quality level together with a different score, requires a different price, so although total welfare and quality is the same for every ex-post equilibrium, the price can be different. This means that total welfare can be divided in a different way between the buyer and the winning seller. Of course, the higher the price, the more this total welfare is distributed towards the winning seller.

Figure 5.3 provides a graphical representation of the fact that maximizing profit by a certain firm i conditional on obtaining a certain score, always leads to the same quality level, but that the resulting price depends on the score to be obtained. This follows from the assumed general forms of the scoring rule and firm $i's$ profit function. Namely, since these are both linear in price, score-lines and profit-lines are obtained from other score-lines and profit-lines by just shifting the initial lines to the right (for lower scores and higher profits) or to the left (for higher scores and lower profits).

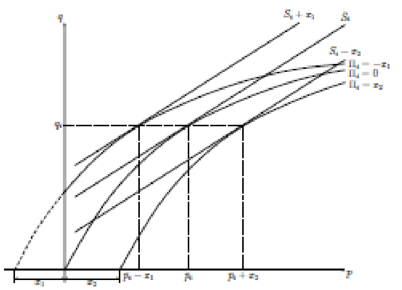

Figure 5.3: Quality does not depend on score when maximizing profit.

The combination which maximizes profit for firm i conditional on obtaining a score of $\bar{S}_i$ (represented by the line in the middle) is $(\bar{p}_i, q_i)$. This leads to 0 profit (represented by the curve in the middle). The combination which maximizes profit for firm i conditional on obtaining a score of $\bar{S}_i + x_1$

(represented by the line on the left, which is obtained by shifting the line in the middle x_1 to the left) is $(\bar{p}_i - x_1, q_i)$. This leads to $-x_1$ profit (represented by the left curve, which is obtained by shifting the curve in the middle x_1 to the left). The combination which maximizes profit for firm i conditional on obtaining a score of $\bar{S}_i - x_2$ (represented by the line on the right, which is obtained by shifting the line in the middle x_2 to the right) is $(\bar{p}_i + x_2, q_i)$. This leads to x_2 profit (represented by the right curve, which is obtained by shifting the curve in the middle x_2 to the right).

5.4.3 Obtaining maximum possible total welfare in first-score auctions if the scoring rule differs from buyer's utility function

Now consider what changes if the scoring rule is not equal to the true utility function of the buyer anymore, so if $S(p,q) \neq U(p,q)$. Commitment to this scoring rule is assumed, so the bid obtaining the highest score will be the winning bid and the buyer cannot change to another scoring rule, for example his utility function, after the bids are submitted. The ex-post equilibria in this setting are the same as the ex-post equilibria in the foregoing analysis, where the scoring rule equaled the buyer's utility function, since the derived necessary and sufficient conditions for an outcome to be an ex-post equilibrium are valid independent of the relation between scoring rule and utility function. To begin with, two examples applied to the FSC-equilibrium are considered.

Let $V(q) = aq$ and $c_i(q) = k_i q^2$, where $a, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. The utility function of the buyer becomes

$$U(p,q) = aq - p$$

and the profit function of any firm i can now be written as

$$\Pi_i(p,q) = p - k_i q^2.$$

Let the scoring rule now be equal to

$$S(p,q) = wq - p,$$

where $w \in \mathbb{R}_+$ represents the weight given to quality in the determination of the score.[7] Firm $\tilde{n}$ will solve

$$\max_{p,q}(wq - p) \text{ subject to } p - \underline{k}_n q^2 = 0 \text{ and } p, q \geq 0.$$

[7] The more general scoring rule looks like $S(p,q) = w_1 q - w_2 p$, where $w_1, w_2 \in \mathbb{R}_+$ represent the weights given to quality and price respectively. To simplify the model, the weight given to price is now normalised (set equal to 1).

By substitution one gets

$$\max_q (wq - \underline{k}_n q^2) \text{ subject to } q \geq 0.$$

Using the first derivative leads to

$$\begin{array}{rcl} w - 2\underline{k}_n q & = & 0 \\ & \Leftrightarrow & \\ q & = & w/(2\underline{k}_n) \end{array}$$

and the second derivative

$$-2\underline{k}_n < 0$$

shows that we are dealing with a maximum. Then $p = \underline{k}_n(w/(2\underline{k}_n))^2 = (1/4)w^2/\underline{k}_n$ and $S = w^2/(2\underline{k}_n) - (1/4)w^2/\underline{k}_n = (1/4)w^2/\underline{k}_n$. Firm m now solves

$$\max_{p,q} (p - \underline{k}_i q^2) \text{ subject to } wq - p = (1/4)w^2/\underline{k}_n \text{ and } p, q \geq 0.$$

Again substituting the first constraint gives

$$\max_q ((wq - (1/4)w^2/\underline{k}_n) - \underline{k}_i q^2) \text{ subject to } q \geq 0.$$

By the first derivative

$$\begin{array}{rcl} w - 2\underline{k}_i q & = & 0 \\ & \Leftrightarrow & \\ q & = & w/(2\underline{k}_i). \end{array}$$

The second derivative

$$-2\underline{k}_i < 0$$

confirms a maximum. This leads to $p = (1/2)w^2/\underline{k}_i - (1/4)w^2/\underline{k}_n$, utility for the buyer now becomes

$$U = (1/2)aw/\underline{k}_i - (1/2)w^2/\underline{k}_i + (1/4)w^2/\underline{k}_n$$

and profit for firm m becomes

$$\begin{array}{rcl} \Pi_m & = & (1/2)w^2/\underline{k}_i - (1/4)w^2/\underline{k}_n - (1/4)w^2/\underline{k}_i \\ & = & (1/4)w^2/\underline{k}_i - (1/4)w^2/\underline{k}_n. \end{array}$$

This implies that total welfare will now equal $(1/2)aw/\underline{k}_i - (1/4)w^2/\underline{k}_i$.

Note that if w equals a (so if the scoring rule would have been equal to the buyer's utility function), this becomes $(1/4)a^2/\underline{k}_i$, which is again the maximum possible total welfare (see the corresponding example in the section about the maximum possible total welfare). However, if $w \neq a$, this amount of total welfare will not be attained. To see this, take the first derivative of the expression for total welfare with respect to w. Doing so yields

$$\begin{aligned} (1/2)a/\underline{k}_i - (1/2)w/\underline{k}_i &= 0 \\ &\Leftrightarrow \\ w &= a \end{aligned}$$

and according to the second derivative

$$-(1/2)/\underline{k}_i < 0$$

this gives a maximum. So total welfare is maximized by setting w equal to a and if $w \neq a$, total welfare will be lower. Next, consider an example in which the form in which quality appears in the utility function (linear) is different from the form in which it appears in the scoring rule (quadratic). Let $V(q) = aq$ and $c_i(q) = k_i q^3$, where $a, k_i \in \mathbb{R}_+$ for all $i \in \{1, 2, ..., N\}$. The utility function of the buyer becomes

$$U(p,q) = aq - p$$

and the profit function of any firm i can now be written as

$$\Pi_i(p,q) = p - k_i q^3.$$

Moreover, let the scoring rule be not linear in quality anymore. To be more precise, consider

$$S(p,q) = wq^2 - p,$$

with $w \in \mathbb{R}_+$. Firm $\tilde{n}$ therefore solves

$$\max_{p,q}(wq^2 - p) \text{ subject to } p - \underline{k}_n q^3 = 0 \text{ and } p, q \geq 0,$$

or equivalently

$$\max_{q}(wq^2 - \underline{k}_n q^3) \text{ subject to } q \geq 0.$$

Setting the first derivative equal to 0 gives

$$2wq - 3\underline{k}_n q^2 = 0$$

$$\begin{array}{rcl} & \Leftrightarrow & \\ q & = & 2w/(3\underline{k}_n) \text{ (or } q=0) \end{array}$$

and the second derivative at this quality level

$$2w - 6\underline{k}_n 2w/(3\underline{k}_n) = -2w < 0$$

guarantees a maximum. Price becomes $p = \underline{k}_n(2w/(3\underline{k}_n))^3 = (8/27)w^3/\underline{k}_n^2$ and this in turn yields $S = (4/9)w^3/\underline{k}_n^2 - (8/27)w^3/\underline{k}_n^2 = (4/27)w^3/\underline{k}_n^2$. This implies for firm m that it has to solve

$$\max_{p,q}(p - \underline{k}_i q^3) \text{ subject to } wq^2 - p = (4/27)w^3/\underline{k}_n^2 \text{ and } p, q \geq 0,$$

which can be simplified to

$$\max_{q}((wq^2 - (4/27)w^3/\underline{k}_n^2) - \underline{k}_i q^3) \text{ subject to } q \geq 0.$$

Solving the maximization problem gives

$$\begin{array}{rcl} 2wq - 3\underline{k}_i q^2 & = & 0 \\ & \Leftrightarrow & \\ q & = & 2w/(3\underline{k}_i) \text{ (or } q=0). \end{array}$$

Then for the second derivative at this quality level, one obtains

$$2w - 6\underline{k}_i 2w/(3\underline{k}_i) = -2w < 0,$$

implying a maximum. Therefore, $p = (4/9)w^3/\underline{k}_i^2 - (4/27)w^3/\underline{k}_n^2$, utility for the buyer

$$U = (2/3)aw/\underline{k}_i - (4/9)w^3/\underline{k}_i^2 + (4/27)w^3/\underline{k}_n^2$$

and profit for firm m

$$\begin{array}{rcl} \Pi_m & = & (4/9)w^3/\underline{k}_i^2 - (4/27)w^3/\underline{k}_n^2 - (8/27)w^3/\underline{k}_i^2 \\ & = & (4/27)w^3/\underline{k}_i^2 - (4/27)w^3/\underline{k}_n^2. \end{array}$$

For total welfare, one will get $(2/3)aw/\underline{k}_i - (8/27)w^3/\underline{k}_i^2$.

There is one value which the constant w can take, which would lead to the maximum possible total welfare. All other values for w would lead to a lower amount of total welfare. To see this more clearly, take the first derivative of total welfare obtained here with respect to w to get

$$(2/3)a/\underline{k}_i - (24/27)w^2/\underline{k}_i^2 \quad = \quad 0$$

$$\Leftrightarrow$$
$$w = \sqrt{(3/4)a\underline{k}_i} \text{ (or } w = -\sqrt{(3/4)a\underline{k}_i}).$$

The second derivative

$$-(48/27)w/\underline{k}_i^2 < 0$$

shows that at this value of w total welfare is maximized. The amount of this maximum total welfare is obtained by filling in this value for w, in the expression for total welfare, which gives

$$(2/3)a(\sqrt{(3/4)a\underline{k}_i})/\underline{k}_i - (8/27)(\sqrt{(3/4)a\underline{k}_i})^3/\underline{k}_i^2.$$

After some rewriting, this becomes

$$a\sqrt{a/(3\underline{k}_i)} - (1/3)a\sqrt{a/(3\underline{k}_i)} = (2/3)a\sqrt{a/(3\underline{k}_i)},$$

which is indeed the maximum possible total welfare (although this expression for the maximum possible total welfare was not derived before, it should not be too difficult for the reader to verify this).

After working through the foregoing examples, it is now time to derive some general condition, by which one could immediately see for which scoring rule(s) (if any), the maximum possible total welfare is obtained in the FSC-equilibrium, even though the scoring rule is different from the buyer's utility function. Assume that the scoring rule will always have the general form

$$S(p, q) = W(q) - p, \tag{5.3}$$

where $W(q)$ can be any increasing function. We now obtain the following proposition of which the proof is given in the complete version of this paper.

Proposition 4. *Consider a first-score auction in which the scoring rule is not equal to the buyer's true utility function. Let this utility function be of the general form in equation* (5.1). *Let the profit function of a firm be of the general form in equation* (5.2). *Let the scoring rule be of the general form in equation* (5.3). *Then the maximum possible total welfare is obtained in the FSC-equilibrium if and only if $W(q) - c_m(q)$ is maximized by the same quality level q as the quality level which leads to this maximum possible total welfare.*

The check the result of the last example, let us apply the general condition of the foregoing proposition to this setting. In this example, $W(q) = wq^2$ and $c_m(q) = \underline{k}_i q^3$. This gives the maximization problem

$$\max_q (wq^2 - \underline{k}_i q^3) \text{ subject to } q \geq 0,$$

which leads to $q = 2w/(3\underline{k}_i)$. Setting this equal to the quality level which leads to the maximum possible total welfare (like the expression for the maximum possible total welfare, also the expression for this quality level was not derived before, but again it should be quite easy for the reader to verify this) yields

$$\begin{array}{rcl} 2w/(3\underline{k}_i) & = & \sqrt{a/(3\underline{k}_i)} \\ & \Leftrightarrow & \\ w & = & \sqrt{(3/4)a\underline{k}_i}, \end{array}$$

which is indeed the value for w obtained before in the example.

Note that in the examples described so far, there turned out to be a unique (feasible) value for w leading to the maximum possible total welfare in the FSC-equilibrium. However, there can exist multiple values for w which would lead to this in some settings, while there may exist no such w in other settings.

After having seen how the results for the FSC-equilibrium change if the scoring rule does not equal the true utility function of the buyer anymore, it is interesting to know how the results change for other ex-post equilibria. Proposition 5 generalizes proposition 4 to these other ex-post equilibria. Again, the proof can be found in the complete version of this paper.

Proposition 5. *Consider a first-score auction in which the scoring rule is not equal to the buyer's true utility function. Let this utility function be of the general form in equation* (5.1)*. Let the profit function of a firm be of the general form in equation* (5.2)*. Let the scoring rule be of the general form in equation* (5.3)*. Then in any ex-post equilibrium, the maximum possible total welfare is obtained if and only if* $W(q) - c_m(q)$ *is maximized by the same quality level as the quality level which leads to this maximum possible total welfare.*

Although the previous results are interesting, one has to realise that there is no good reason to believe that the scoring rule is indeed chosen such that the derived condition holds. In fact, the derived condition is important for a social planner (who wants to obtain the maximum possible total welfare) if he can design the scoring rule, but in practice the buyer (who wants to maximize his own utility) usually chooses the scoring rule. To see how the scoring rule then differs from the scoring rule which maximizes total welfare and which implications this has, one example will be provided. For other cases, of course the same idea can be applied and similar results can be obtained.

Remember the first example given for the FSC-equilibrium in which the scoring rule differed from the utility function of the buyer. Total welfare was found to be given by $(1/2)aw/\underline{k}_i - (1/4)w^2/\underline{k}_i$, which was maximized by setting w equal to a. Buyer's utility equaled

$$U = (1/2)aw/\underline{k}_i - (1/2)w^2/\underline{k}_i + (1/4)w^2/\underline{k}_n.$$

Taking the first derivative with respect to w of this expression, instead of the expression for total welfare gives

$$\begin{array}{rcl}
(1/2)a/\underline{k}_i - w/\underline{k}_i + (1/2)w/\underline{k}_n & = & 0 \\
 & \Leftrightarrow & \\
(1/2)a/(w\underline{k}_i) - 1/\underline{k}_i + 1/(2\underline{k}_n) & = & 0 \\
 & \Leftrightarrow & \\
(1/2)a/(w\underline{k}_i) & = & 1/\underline{k}_i - 1/(2\underline{k}_n) \\
 & = & (2\underline{k}_n)/(2\underline{k}_n\underline{k}_i) - \underline{k}_i/(2\underline{k}_n\underline{k}_i) \\
 & = & (2\underline{k}_n - \underline{k}_i)/(2\underline{k}_n\underline{k}_i) \\
 & \Leftrightarrow & \\
2w\underline{k}_i/a & = & (2\underline{k}_n\underline{k}_i)/(2\underline{k}_n - \underline{k}_i) \\
 & \Leftrightarrow & \\
w & = & (a\underline{k}_n)/(2\underline{k}_n - \underline{k}_i) \\
 & < & a.
\end{array}$$

The last step follows from $\underline{k}_i < \underline{k}_n$, which implies $\underline{k}_n/(2\underline{k}_n - \underline{k}_i) < 1$. This means that the weight given to quality to obtain the maximum possible total welfare was too high in the eyes of the buyer. Buyer's utility is indeed maximized by the obtained w, which is confirmed by the second derivative

$$-1/\underline{k}_i + 1/(2\underline{k}_n) < 0.$$

Moreover, observe that since $\underline{k}_i > 0$, $\underline{k}_n/(2\underline{k}_n - \underline{k}_i) > 1/2$. Filling in the derived w in the expression for total welfare leads to

$$\begin{array}{rcl}
TW & = & a/(2\underline{k}_i)((a\underline{k}_n)/(2\underline{k}_n - \underline{k}_i)) - 1/(4\underline{k}_i)((a\underline{k}_n)/(2\underline{k}_n - \underline{k}_i))^2 \\
 & = & a^2/(2\underline{k}_i)(\underline{k}_n/(2\underline{k}_n - \underline{k}_i)) - a^2/(4\underline{k}_i)(\underline{k}_n/(2\underline{k}_n - \underline{k}_i))^2 \\
 & = & a^2/(4\underline{k}_i)((2\underline{k}_n/(2\underline{k}_n - \underline{k}_i)) - (\underline{k}_n/(2\underline{k}_n - \underline{k}_i))^2).
\end{array}$$

Because $1/2 < \underline{k}_n/(2\underline{k}_n - \underline{k}_i) < 1$, this will always be lower than $a^2/(4\underline{k}_i)$, which is the maximum possible total welfare (see the corresponding example in the section about the maximum possible total welfare). In the best case, $\underline{k}_i$ is 'very close to' $\underline{k}_n$, which makes $\underline{k}_n/(2\underline{k}_n - \underline{k}_i)$ 'very close to' 1 and total welfare 'very close to' the maximum possible total welfare. In the worst case, $\underline{k}_i$ is 'very close to' 0, which makes $\underline{k}_n/(2\underline{k}_n - \underline{k}_i)$ 'very close to' $1/2$ and total welfare approximately 75% of the maximum possible total welfare.

Note that if the most efficient firm would be able to determine the scoring rule (still satisfying the same general form), it would choose weight w as high as possible. Namely, again since $\underline{k}_i < \underline{k}_n$, the expression for this firm's profit

$$\Pi_m = (1/4)w^2/\underline{k}_i - (1/4)w^2/\underline{k}_n$$

is higher, the higher w. It is in the benefit of society that the most efficient firm is never able to choose the scoring rule, since if $w \to \infty$, total welfare $(1/2)aw/\underline{k}_i - (1/4)w^2/\underline{k}_i \to -\infty$.

5.5 Conclusion

After introducing the general concepts of multidimensional procurement auctions and scoring rules (section 1), the performance of such multidimensional auctions was compared to the performance of alternative award mechanisms (one-dimensional auctions and negotiations) by evaluating their theoretical advantages and disadvantages (section 2). For every mechanism, there exist arguments in favour of and against its use. Which mechanism performs the best in practice seems to depend heavily on the specific characteristics of the procurement market at hand. Multidimensional auctions, one-dimensional auctions and negotiations are all frequently used, but less seems to be known about multidimensional auctions than about the other mechanisms. This can be explained by the fact that multidimensional auctions are harder to analyse and more complex to organise. The main objective of the analysis in section 3 was to derive the maximum possible total welfare resulting from a multidimensional procurement auction for several combinations of utility functions of buyers and profit and cost functions of sellers. Afterwards, in section 4, equilibria were derived for first-score auctions and conditions were obtained under which the maximum possible total welfare is achieved in these equilibria.[8] Distinction was made between the case in which the scoring rule equals the utility function of the buyer and the case in which these are different from each other.

Just like the advantages and disadvantages of award mechanisms mentioned in section 2 were mainly of theoretical importance, it is important to realise that also the derived equilibria are probably only interesting from

[8] In the complete version of this paper, similar analyses are presented for second-score and second-preferred-offer auctions. In contrast to first-score and second-preferred-offer auctions, in second-score auctions there turned out to exist not only ex-post equilibria, but also a dominant strategy equilibrium.

a theoretical perspective. It can in no way be guaranteed that the outcome of an auction indeed forms an equilibrium. This can have several reasons. Bidders may simply not know which bid to submit in order to be in equilibrium or they might change their bid based on expectations on bids submitted by other bidders. It is for example interesting to know that the maximum possible total welfare is always obtained in an ex-post equilibrium of a first-score auction in which the scoring rule equals the buyer's utility function (proposition 3), but in practice the maximum possible total welfare is not always obtained in a first-score auction in which the scoring rule equals the buyer's utility function, precisely because the submitted bids may fail to form an ex-post equilibrium. Especially if bidders are unexperienced, it can be expected that they do not know which bidding strategy performs the best for them. Moreover, if auctions consist of only one period, it is impossible to apply what one has learned from a previous bidding round, while if auctions last for several rounds, bidders can adapt their bids based on new information and will form new expectations, which could lead to convergence to equilibrium.

In order to be able to form reasonable and appropriate expectations on the outcome of a certain auction, economic experiments can be performed before the actual auction takes place. In such experiments, (almost) all characteristics of a procurement auction which play an important role (think for example of the auction type, the number of bidders, the item to be procured and the aspects to be specified in a bid) can be controlled. This means that the real life setting can be replicated. Buyers can then observe how bidders will behave under different circumstances and the bidders themselves can learn which bidding strategy works the best based on bids from competing bidders. Economic experiments have already more than once proven to be useful in designing procurement markets and auctions in general.

Although quite some research is already done in the field of multidimensional auctions and scoring rules, there are still multiple questions left to be answered. Therefore, there remains a large need for further research. Not only experimental, but also theoretical research is not completed yet. Often it is assumed that all bidders submit one sealed bid and that the auction then ends already, so the focus is on one-shot auctions. However, like is the case for one-dimensional auctions, also multidimensional auctions can last for multiple rounds. Furthermore, one could for example think of multidimensional versions of ascending auctions (like the English auction) and descending auctions (like the Dutch auction). Research concerning the performance of such types forms an example of possible future research. Also the research addressing non-quasi-linear scoring rules should be extended. The general made assumption of quasi-linear scoring rules is namely quite restrictive. Besides, it is worth mentioning that the assumption that cost functions of firms are publicly known is restrictive (and not realistic) as well. Most often the assumption is therefore made that cost functions are privately observed. Although it is then not immediately clear in the market anymore which firm is the most efficient one, which is normally also the case in real life, it is still assumed that there is one firm with the lowest costs for delivering any quality level. This implies that the most efficient firm at a quality level of 5 is also the most efficient firm at a quality level of 10. This is not impossible to be the case in practice, but this is not necessary and always the case. It is for example possible that a certain firm is the most efficient firm for a quality level up to 5, while another firm is most efficient for a quality level between 5 and 10 and yet another firm is most efficient for a quality level above 10. Such a situation forms another interesting subject to be analysed in future research. All in all, it should be clear that although some research has already been done, there is still need and room for further research in the field of multidimensional auctions and scoring rules.

Bibliography

[Bulow and Klemperer(1996)] Bulow, J. and Klemperer, P. (1996). Auctions Versus Negotiations. *The American Economic Review, 86*(1), pp. 180 - 194.

[Burguet and Che(2004)] Burguet, R. and Che, Y. (2004). Competitive Procurement with Corruption. *RAND Journal of Economics, 35*(1), pp. 50 - 68.

[Cabizza and De Fraja(1998)] Cabizza, M. and De Fraja, G. (1998). Quality Considerations in Auctions for Television Franchises. *Information Economics and Policy, 10*, pp. 9 - 22.

[Crémer and McLean(1985)] Crémer, J. and McLean, R. (1985). Optimal Selling Strategies under Uncertainty for a Discriminating Monopolist when Demands are Interdependent. *Econometrica, 53*(2), pp. 345 - 361.

[Estache et al.(2009)] Estache, A., Guasch, J., Iimi, A. and Trujillo, L. (2009). Multidimensionality and Renegotiation: Evidence from Transport-Sector Public-Private-Partnership Transactions in Latin America. *Review of Industrial Organization, 35*(1), pp. 41 - 71.

[Goldberg(1977)] Goldberg, V. (1977). Competitive Bidding and the Production of Precontract Information. *The Bell Journal of Economics, 8*(1), pp. 250 - 261.

[Manelli and Vincent(1995)] Manelli, A. and Vincent, D. (1995). Optimal Procurement Mechanisms. *Econometrica, 63*(3), pp. 591 - 620.

[Milgrom and Roberts(1992)] Milgrom, P. and Roberts, J. (1992). *Economics, Organization and Management* (1st ed.). London: Prentice Hall.