**Summary**

The starting point of the present paper is the *Binomial Option Pricing Model*. It basically assumes that thers is only one possible value for the volatility of the stock price and it gives a unique arbitrage-free price of an option. This assumption is relaxed in the sense that the occurrence of a second value for the volatility is supposed to have strictly positive probability. Then it is no longer possible to find only one arbitrage-free price of the option; instead some concepts of general arbitrage pricing theory such as the *Fundamental Theorem of Asset Pricing* are employed to construct an interval of arbitrage-free option prices. Subsequently, a natural question to ask is under which conditions the *Binomial Option Pricing Model* assuming deterministic and stochastic volatility respectively agrees on arbitrage-free option prices. This paper gives a formal answer to that question by showing that the volatility used to calculate the price in the deterministic setting has to lie strictly in between the two possible volatilities used when assuming stochastic volatility. Furthermore, the *Binomial Option Pricing Model* is enriched to the *Trinomial Option Pricing Model*. In the latter model in cases of both deterministic as well as stochastic volatility an interval of arbitrage-free option prices is obtained. The question of agreement on arbitrage-free prices is discussed again and a similar answer to the above situation is derived. The paper provides illustrations of the theoretical findings along examples such as European Call Options, Butterfly Spreads as well as Double Butterfly Spreads.

## 2.1 Introduction

> A normal person and a professor working at the Mathematical Finance department take a walk. The normal person suddenly sees a 100€ bill lying around on the street. When the normal person wants to pick it up, the professor says:"Don't try to do that. It is absolutely impossible that a 100€ bill is lying on the street. Indeed, if it were lying on the street, somebody else would have come and picked it up before you"
>
> [4]

In *Financial Mathematics* various concepts are devoted to the determination of asset prices. Derivative securities, which derive their value from underlying more basic assets, play an important role in this regard. A stock is a typical example of an asset that is underlying a derivative security. Presumably one of the most well-known kinds of a derivative securitiy is an option such as a call or a put. In case of a European call or put option, its value is completely determined by the underlying's price at the time of the option's expiration. Hence, in order to determine the price of the option at the time it is written, some knowledge regarding the price of the stock at maturity is desirable. This is the kernel of the problem: Of course noone knows the value of a stock at some future date with certainty. For that reason mathematical tools and models are used to gain knowledge of the future share price. To this end, some assumptions have to be made in order to systematize things. An example of such an assumption is how many different values the stock price can possibly assume at the time of the derivative's expiration. Other assumptions refer to the market in which the assets are traded. A natural question to ask is then what is going to happen if one (or several) assumptions are modified. Clearly, the results given by the respective models are going to change. It is seemingly difficult to decide which assumptions mirror reality best, thus it is of utmost interest if different models give similar results - this is the central question of the thesis at hand. It focuses on the *Binomial* as well as the *Trinomial* model for asset pricing and explores what happens to the results if one allows for more than one volatility. Particular attention is paid to the circumstances under which the respective models agree on arbitrage-free prices.

The paper is organized as follows: Section 2 introduces some mathematical concepts that are relevant in the theory of Asset Pricing. Section 3 firstly derives the *Binomial Asset Pricing Model* with one possible value of the volatility and then discusses the case of a second possible value. These sections will be mainly theoretical in nature. In order to illustrate the concepts a bit further, section 4 provides several numerical examples and graphs. It will be outlined that there is some general pattern regarding agreement and disagreement on arbitrage-free prices throughout all the examples described, thus section 5 systematizes this pattern and provides a proof. Section 6 enriches the *Binomial Asset Pricing Model* to the *Trinomial Asset Pricing Model*; again the focus is first on the case of one volatility and then it will be allowed for a second value. Here a lot of theory developed earlier can be applied. Similar to section 4, section 7 provides various numerical examples. Readers will see that there is again a general pattern regarding the no-arbitrage prices when comparing results in cases of deterministic as well as stochastic volatility, thus section 8 discusses a general case. Finally, section 9 summarizes and concludes.

## 2.2 Background Information

### 2.2.1 The No-Arbitrage Assumption

When referring to an arbitrage opportunity, one usually means the possibility to gain a profit in financial markets with neither taking any risk nor undertaking a net investment [4]. The "joke" at the beginning of this paper states the essence of what is called *Pricing by No-Arbitrage*. The professor working in the Mathematical Finance department knows that it is not possible to simply find 100€ lying around since this would mean a riskless profit without net investment. If arbitrage opportunities were possible, everyone would seek to exploit them as soon as possible, therefore market forces would make them disappear immediately and bring the market back to equilibrium. This is the reason why the professor says that someone else would have already picked up the 100€ bill. Hence, assuming the abscence of arbitrage opportunities when pricing financial assets seems reasonable. In order to conceptualize things a bit more, consider the finite filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Here $\Omega$ is the set of all possible future states of nature and assume $|\Omega| < \infty$. Let $\omega$ be a typical element of $\Omega$ and suppose that each future state of the world is assigned a strictly positive probability of occurrence, i.e. $\mathbb{P}(\omega) > 0$. Let $\Phi$ denote the linear space of all portfolios containing primary traded securities only. Suppose $\phi$ is a typical element of $\Phi$. Considering $V_0(\phi)$ as the cost of setting up the portfolio and letting $T$ be some later point in time, the idea of *No-Arbitrage* can be formally stated as follows [6].

**Definition 1.** *A security pricing model is said to be arbitrage-free if there is no portfolio $\phi \in \Phi$ for which*

$$V_0(\phi) = 0,\ V_T(\phi) \geq 0 \text{ and } \mathbb{P}[V_T(\phi) > 0] > 0 \tag{2.1}$$

The *No-Arbitrage* assumption is maintained throughout this paper.

### 2.2.2 Complete and Incomplete Markets

The phrase "state of the world" simply means a possible outcome of a certain process: Consider for instance a share that can assume two possible values on the next day - then each of these values constitutes a state of the world. When referring to future states of the world, the time horizon is to be specified; since this paper is focussing on the pricing of derivatives, the expiry date of such an asset will make up the time horizon. The value of a derivative security at expiration is of course contingent on the state of occurrence. This gives rise to the following definition taken from [1].

**Definition 1.** *A contingent claim*

As the payoff of a contingent claim depends on the future states of nature and since from today's point of view it is unknown what state is going to be realized, the determination of the price of a contingent claim is relatively difficult. Though there are methods on how to determine a unique price - these methods will be elaborated on in later chapters -, the possibilities of pricing a contingent claim strongly depend on the market setting. The following definition of *Complete Markets* is taken from [1] (here $\mathbb{F}_T$ is meant to be an element of the filtration, i.e. $\mathbb{F}_T \subseteq \mathbb{F}$).

**Definition 2.** *A market is complete if every contingent claim is attainable, i.e. for every $\mathbb{F}_T$-measurable random variable $A$ there exists a replicating self-financing portfolio $\phi \in \Phi$ such that $V_T(\phi) = A$.*

In an *Incomplete Market* it is obviously not possible to find a replicating strategy such that the cash flows of the contingent claim at maturity can be reached.

### 2.2.3 The Fundamental Theorem of Asset Pricing

Consider again the finite filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. Let $S$ denote a financial market modeled on this space (the interested reader may refer to [4] for a detailed elaboration on financial markets in general). A lot of theory in this paper is based on the concept of an *equivalent martingale measure* which is formally explained in the following definition taken from [4].

**Definition 3.** *A probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{F})$ is called an equivalent martingale measure for $S$, if $\mathbb{Q} \sim \mathbb{P}$ and $S$ is a martingale under $\mathbb{Q}$. i.e. $\mathbb{E}_{\mathbb{Q}}[S_{t+1}|\mathbb{F}_t] = S_t$ for $t = 0, \cdots, T-1$.*

In analogy to the just mentioned two authors, define $M^e(S)$ to be the set of all equivalent martingale measures and denote by $M^a(S)$ the set of all martingale measures. Note that the latter are not necessarily equivalent to $\mathbb{P}$. Before going on, one should reflect a moment on what is meant by *equivalent*. It has been previously defined that $\forall \omega \in \Omega$ it holds that $\mathbb{P}(\omega) > 0$. Thus $\mathbb{Q} \sim \mathbb{P}$ if and only if $\mathbb{Q}(\omega) > 0 \ \forall \omega \in \Omega$ [4]. Keeping this concept in mind, one can go on to one of the most essential Theorems in the field on Financial Mathematics. *The Fundamental Theorem of Asset Pricing*, Theorem 1 in this paper, is linking the concepts discussed so far, i.e. the *No-Arbitrage* ideas with the theory on martingales, and it is therefore a powerful tool for determining prices.

**Theorem 1. (Fundamental Theorem of Asset Pricing)** *For $S$ modeled on $(\Omega, \mathbb{F}, \mathbb{P})$, the following two statements are equivalent:*
*(i) $S$ satisfies the No-Arbitrage condition*
*(ii) $M^e(S) \neq \emptyset$*

*Proof.* See [4] □

## 2.3 The Binomial Model

The Binomial Model for Option Pricing has been presented first by [3]. The basic idea underlying this approach is the replication of a derivative's payoff using the marketed assets. Before discussing the Binomial Tree approach, there are some assumptions that have to be made. Since the matter of interest in this paper is the price of options in complete and incomplete markets, it is tried to keep most things as simple as possible. To this end, the attention is restricted to a two-period economy, let *0* and *T*=1 be the two points in time such that $\Delta t = 1$. Moreover, suppose there are only two traded assets in this economy: A bond with risk-free rate of return $r_f$ and a stock with initial price $S_0$ and two possible prices at time $1$. Furthermore, it is assumed that the shares of the stock can be subdivided for purchase and sale, the interest rate for investing and borrowing is the same, namely $r_f$, and there is no bid-ask spread for purchasing and selling the stock. [7].

### Notational Remark

In total four models will be presented in this thesis. To make things as clear as possible, parameters that occur in each model, e.g. the stock price, will carry the numeration of the model (1 - 4) as superscript. Subscripts refer to the respective states.

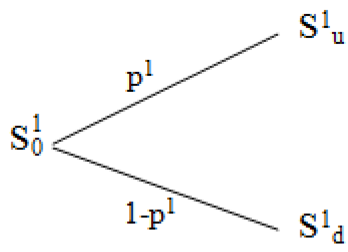


Figure 2.1: Binomial Tree

### 2.3.1 Deterministic Volatility

#### Derivation

Consider the finite filtered probability space $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1)$ and let $S_0^1$ denote the initial stock price. As there are only two values of the share price possible at time $1$, it follows that $|\Omega^1| = 2$. Suppose the values the stock can assume at time $1$ are given by $S_u^1 = S_0^1 \cdot u$ and $S_d^1 = S_0^1 \cdot d$ with $u > d$. In the remainder of the paper $u$ is referred to as the *up factor* and $d$ is called the *down factor*. Obviously, in the Binomial setting the stock price has to either increase or decrease until time $1$ with respect to time $0$, there is no possibility that it remains constant. One usually imposes that $d = \frac{1}{u}$. Define $u = e^{\sigma}$ for some appropriate value $\sigma$ that is henceforth referred to as the volatility; the definition comes from the approximation to the Geometric Brownian Motion [3]. In order to not allow for arbitrage opportunities, the following inequality must hold:

$$0 < d < 1 + r_f < u \tag{2.2}$$

Indeed, [7] shows that in the one-period model there is no arbitrage opportunity if and only if (2.2) holds.

Let $p^1 \in (0, 1)$ denote the probability of an upward move of the stock price; then $1-p^1$ is the probability of a downward move. Figure 2.1 depicts the situation just described.

In the following a European call option is considered. This option enables the option holder to buy one share of a stock at time $1$ at a predefined strike price $K$. Note that the option gives the holder the right and not the obligation to buy the share. For that reason the holder of course does not exercise the option if the strike price is above the stock price. Letting $C_1^1$ denote the call's payoff, this implies that $C_1^1 = max(S_1^1 - K, 0)$. In light of the above definition, the option is obviously a contingent claim dependent on the value the stock assumes at time $1$. The question of interest now is the determination of the call price at time $0$ denoted by $C_0^1$. The arbitrage pricing theory idea is the replication of the option's payoff using the stock and the bond - this is possible because the market is complete. The following derivation is mainly based on [7].

Suppose an agent in the economy at hand possesses initial wealth $W_0$ and she buys $\Delta_0$ shares of the stock. Her leftover in terms of cash at time $0$ is then given by

$$M_0 = W_0 - \Delta_0 \cdot S_0^1 \tag{2.3}$$

At time $1$, the cash position is then given by

$$M_1 = \Delta_0 \cdot S_1^1 + e^{r_f} \cdot (W_0 - \Delta_0 \cdot S_0^1) = e^{r_f} \cdot W_0 + \Delta_0 \cdot (S_1^1 - e^{r_f} \cdot S_0^1) \tag{2.4}$$

How does the replication of the call work now? There are two possible values for the call payoff at time $1$ that are denoted respectively by $C_1^1(u)$ and $C_1^1(d)$. The replicating portfolio is supposed to yield the same payoff. This inevitably leads to the following two replication conditions:

$$M_1(u) = C_1^1(u) \tag{2.5}$$

$$M_1(d) = C_1^1(d) \tag{2.6}$$

Plugging in the corresponding values and discounting back to time point $0$, the equations become

$$W_0 + \Delta_0 \cdot (e^{-r_f} \cdot S_1^1(u) - S_0^1) = e^{-r_f} \cdot C_1^1(u) \tag{2.7}$$

$$W_0 + \Delta_0 \cdot (e^{-r_f} \cdot S_1^1(d) - S_0^1) = e^{-r_f} \cdot C_1^1(d) \tag{2.8}$$

Let $q^1 \in (0,1)$ and multiply (2.7) and (2.8) by respectively $q^1$ and $(1-q^1)$; this yields

$$W_0 \cdot q^1 + \Delta_0 \cdot (e^{-r_f} \cdot S_1^1(u) - S_0^1) \cdot q^1 = e^{-r_f} \cdot C_1^1(u) \cdot q \tag{2.9}$$

$$W_0 \cdot (1-q^1) + \Delta_0 \cdot (e^{-r_f} \cdot S_1^1(d) - S_0^1) \cdot (1-q^1) = e^{-r_f} \cdot C_1^1(d) \cdot (1-q^1) \tag{2.10}$$

Add up the previous two equations to obtain

$$W_0 + \Delta_0 \cdot (e^{-r_f} \cdot (S_1^1(u) \cdot q^1 + S_1^1(d) \cdot (1 - q^1)) - S_0^1) \tag{2.11}$$
$$= e^{-r_f} \cdot (q^1 \cdot C_1^1(u) + (1 - q^1) \cdot C_1^1(d)) \tag{2.12}$$

Choosing $q^1$ in such a way that

$$S_0^1 = e^{-r_f} \cdot (S_1^1(u) \cdot q^1 + S_1^1(d) \cdot (1 - q^1)) \tag{2.13}$$

(2.10) reduces to

$$W_0 = e^{-r_f} \cdot (q^1 \cdot S_1^1(u) + (1 - q^1) \cdot S_1^1(d)). \tag{2.14}$$

Then $q^1$ can be expressed as follows:

$$q^1 = \frac{e^{r_f} - d}{u - d} \tag{2.15}$$

Finally, $\Delta_0$ is given by:

$$\Delta_0 = \frac{C_1^1(u) - C_1^1(d)}{S_1^1(u) - S_1^1(d)} \tag{2.16}$$

To sum up, the replication of the call works as follows: The agent possessing the initial wealth given by (2.14) buys the amount of shares given by (2.16) at time $0$. Regardless which state of nature, i.e. up or down, is realized at time $1$, the replicating portfolio will have the respective value, that is either $C_1^1(u)$ or $C_1^1(d)$. This means that the agent has hedged the short position in the derivative [7]. Hence, the call price at time $0$, $C_0^1$, is to obey the following price:

$$C_0^1 = e^{-r_f} \cdot (q^1 \cdot C_1^1(u) + (1 - q^1) \cdot C_1^1(d)) \tag{2.17}$$

It is worthwhile to have a closer look at the $q^1$ as given by (2.15). Note that (2.2) implies that $q^1$ is larger than zero. It has been multiplied by respectively $q^1$ and $(1 - q^1)$. Since their sum is one and the fact that $q^1$ is positive suggests to consider it as a probability. Of course it is not the actual probability that one particular state occurs as shown on the branches in Figure 2.1, but it is called the *risk-neutral probability* [7]. Hence one may refer to (2.17) as the *risk-neutral valuation formula*. Note also that the actual probabilities do not appear in this equation.

At this stage it seems reasonable to link the above derivation to the theoretical concepts introduced in section 2. In equation (2.13), $q^1$ and $(1-q^1)$ fulfill the definition of a martingale measure for the stock price process. Since the two probabilities are by definition strictly between $0$ and $1$, they are equivalent to the actual probabilities leading to the upward and downward movement of the stock price. In view of the *Fundamental Theorem of Asset Pricing*, this implies that there are no arbitrage opportunities in the market. Under the martingale measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}^1$, the discounted expected call option payoff at time $1$ yields an arbitrage-free price at time $0$. Hence, (2.17) can be restated as

$$e^{-r_f} \cdot \mathbb{E}_{Q^1}[C_1^1] = C_0^1 \tag{2.18}$$

Note that the probability measure $\mathbb{Q}^1$ is also referred to as *risk-neutral measure*. For simplicity's sake the remainder of this paper assumes $r_f = 0$.

**Internal Summary**

The idea just discussed is the following: The call option value at time $1$ can be replicated by trading the stock and the bond, so its value is a linear function in the span of the stock price and the bond price at time $1$. For that reason the value of the call at time $0$ should be the corresponding linear combination of the stock and the bond price at time $0$. Since the no-arbitrage condition implies the existence of a martingale measure under which the stock price process is a martingale (and the bond price of course as well), this measure can be used to price the call.

### 2.3.2 Stochastic Volatility

Although the discussion in the previous section has been subsumed under the tile "Deterministic Volatility", not much emphasis has been lain on volatility. So far, it has been assumed that there is one and only one volatility $\sigma$ of the stock price. Consider in the following the finite filtered probability space $(\Omega^2, \mathbb{F}^2, \mathbb{P}^2)$ and assume that for $\alpha \in (0,1)$, $\mathbb{P}^2(\sigma = \sigma_1) = \alpha$ and $\mathbb{P}^2(\sigma = \sigma_2) = 1 - \alpha$. Moreover, suppose without loss of generality $\sigma_1 < \sigma_2$. Under this assumption the volatility is obviously no longer deterministic but stochastic. In the binomial model discussed so far, this means that at time $1$ there are four possible values the share price can assume, namely

1. $S_0^2 \cdot e^{\sigma_1}$
2. $S_0^2 \cdot e^{-\sigma_1}$
3. $S_0^2 \cdot e^{\sigma_2}$
4. $S_0^2 \cdot e^{-\sigma_2}$

It is still the aim to price the European call with strike price $K$. As $|\Omega^2| = 4$ and since there are only two marketed assets, it is no longer possible to entirely replicate the payoff pattern of the call at time $1$, therefore the market is incomplete now. Thus, when pricing the call, one does no longer obtain a unique price.

### The no-arbitrage boundaries

The main question now is the description of the possible call prices. A reasonable way to do so seems to be the application of the Arbitrage Pricing Theory discussed previously. To this end, define $S^2$ as the stock price process under the stochastic volatility and let $\mathbb{Q}^2$ be a martingale measure equivalent to the actual probability measure $\mathbb{P}^2$ such that $S^2$ is a martingale under $\mathbb{Q}^2$. In the following, $q_j^2$ is referred to as the risk-neutral probability leading to state j, $0 \leq j \leq 4$. Adopting the approach used in the previous section, an arbitrage-free call price at time $0$, here denoted by $C_0^2$, is the expectation of the call's payoff at time $1$ under $\mathbb{Q}^2$. Let $c_j^2$ denote the call's payoff in state j. Then

$$W := \{\sum_{j=1}^{4} q_j^2 \cdot c_j^2 | \sum_{j=1}^{4} q_j^2 = 1 \textit{ and } \sum_{j=1}^{4} q_j^2 \cdot s_j^2 = S_0^2 \textit{ where } q_j^2 > 0\} \quad (2.19)$$

describes the set of all possible arbitrage-free prices of the call. Note that $W$ is a subset of $\mathbb{R}$. It is straightforward to show that $W$ is convex: Let $\mathbb{Q}_a^2$ and $\mathbb{Q}_b^2$ be two martingale measures equivalent to $\mathbb{P}^2$ such that

$$\mathbb{E}_{\mathbb{Q}_a^2}[S^2] = S_0^2 \quad (2.20)$$

$$\mathbb{E}_{\mathbb{Q}_b^2}[S^2] = S_0^2 \quad (2.21)$$

and note that for $\lambda \in (0, 1)$ the following statement holds true:

$$\mathbb{E}_{\lambda\mathbb{Q}^2_a+(1-\lambda)\mathbb{Q}^2_b}[S^2] = \lambda \cdot S^2_0 + (1-\lambda) \cdot S^2_0 = S^2_0 \tag{2.22}$$

For that reason this convex combination is an equivalent martingale measure yielding an arbitrage-free call price as well. Since an entire replication of the call's payoff is not possible in this market setting, $W$ is not a singleton set. In order to get a more precise idea of what $W$ looks like, the following lemma might be useful.

**Lemma 1.** *A bounded convex set $D$ with $|D| > 1$ in $\mathbb{R}$ is an interval.*

*Proof.* Let D be a convex set in $\mathbb{R}$. Define $a := \inf D$ and $b := \sup D$. Hence to show: $D \in \{(a,b), [a,b), (a,b], [a,b]\}$. The latter statement holds true if and only if $(a,b) \subseteq D \subseteq [a,b]$, In order to show the first inclusion, let $c \in (a,b)$. According to the definition of $a$ and $b$ there are $d, e \in D$ such that $a \leq d < c$ and $c < e \leq b$. Since D is convex, all numbers in between $d$ and $e$ must be contained in $D$, thus $c \in D$. Note that the inclusion $D \subseteq [a,b]$ is true by the definition of $a$ and $b$. □

Lemma 1 obviously implies that $W$ is an interval. [4] show that the boundaries are not contained in the interval, hence it is open. Interested readers may refer to pp.24-25 of their book for a detailed discussion. In order to find the boundaries of the interval for the no-arbitrage price of the call option, it is convenient to formulate this task as an optimization problem with constraints. The lower (upper) bound of the interval is obtained by solving the following minimization (maximization) problem:

$$\begin{aligned} \text{max/min} \quad & \sum_{\text{j}=1}^{4} q^2_j \cdot c^2_j \\ \text{subject to} \quad & \sum_{\text{j}=1}^{4} q^2_j = 1 \\ & \sum_{\text{j}=1}^{4} q^2_j \cdot s^2_j = S^2_0 \\ & q^2_j > 0 \ \ \forall j \end{aligned} \tag{2.23}$$

### Notational Remark

In the remainder of this paper, the model introduced in 2.3.1 is referred to as model I and the one introduced in the current section, 2.3.2, is named model II.

## 2.4 Numerical Examples I

In order to get some further impressions of the two models discussed so far, it is convenient to look at some numerical examples. The option price will be plotted in terms of the volatility, therefore the impacts of deterministic and stochastic volatility are nicely illustrated.

### 2.4.1 European Call

The two models are now used to price a European call option. In model I this is straightforwardly done; denote the price of a call option as a function of the volatility in model I at time $0$ by $C_0^1(\sigma)$. Using equation (2.17) and the risk-free probability as defined by (2.15), the following function can be derived:

$$C_0^1(\sigma) = \frac{1-e^{-\sigma}}{e^{\sigma}-e^{-\sigma}} \cdot max(S_0^1 \cdot e^{\sigma} - K, 0) + \frac{e^{\sigma}-1}{e^{\sigma}-e^{-\sigma}} \cdot max(S_0^1 \cdot e^{-\sigma} - K, 0) \tag{2.24}$$

Note that this function is increasing in the volatility. Consider the following values for the parameters:

- $S_0^1 = S_0^2 = 10$
- K=10
- Possible volatilities in model II: $\mathbb{P}^2(\sigma = 0.15) = \alpha$ and $\mathbb{P}^2(\sigma = 0.20) = 1 - \alpha$ for some $\alpha \in (0, 1)$

The situation is graphically depicted in Figure 2.2. The numbers in braces below the stock value in the respective state refer to the corresponding call option payoff.

In order to find the no-arbitrage boundaries of model II, the numbers have to be plugged in such that the following optimization problem is faced according to (2.23).

$$\begin{aligned} \text{max/min} \quad & 1.6183q_1^2 + 0q_2^2 + 2.2140q_3^2 + 0q_4^2 \\ \text{subject to} \quad & \sum_{j=1}^{4} q_j^2 = 1 \\ & 11.6183q_1^2 + 8.6071q_2^2 + 12.2140q_3^2 + 8.1873q_4^2 = 10 \\ & q_j^2 > 0 \ \ \forall j \end{aligned} \tag{2.25}$$

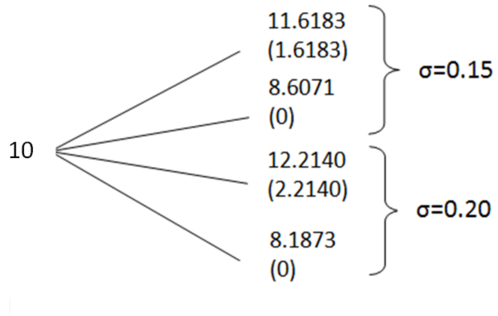


Figure 2.2: Four States

Denote the solutions by respectively $C_d^2$ and $C_u^2$. This problem has been solved with the help of Aimms 3.12 and the following values have been obtained: $C_d^2 = 0.7485824851$ and $C_u^2 = 0.9966761345$. All values in between the two no-arbitrage boundaries are arbitrage-free prices of the call in model II. Now it is interesting to investigate the relation between arbitrage-free prices in model I and model II. Plugging $S_0^1 = 10$ and K=10 into (2.24), the formula for the option price in model I in this example boils down to

$$C_0^1(\sigma) = 10 \cdot \frac{1 - e^{-\sigma}}{e^{\sigma} - e^{-\sigma}} \cdot (e^{\sigma} - 1) \tag{2.26}$$

In Figure 2.3 the horizontal axis measures the volatility and the vertical axis corresponds to the option value. The two boundaries obtained by solving (2.25) have been plotted as horizontals and (2.26) is represented by the ascending line. Obviously both models agree on an arbitrage-free price only for an interval of volatilities. Looking at the volatilities corresponding to that interval, it seems that the intersection of the line representing the option price in model I intersects the two horizontals at volatilities of respectively 0.15 and 0.20. Indeed, (2.26) yields $C_0^1(0.15) = 0.74859691$ and $C_0^1(0.2) = 0.99667995$. These values differ from the ones obtained by solving (2.25) only after several digits (that might be due to calculations performed with rounded values). However, this observation leads to the question whether it is in general the case that a price is arbitrage-free in model I as well as model II only if the underlying volatility in model I is between $\sigma_1$ and $\sigma_2$. The next section provides further insight on that issue.

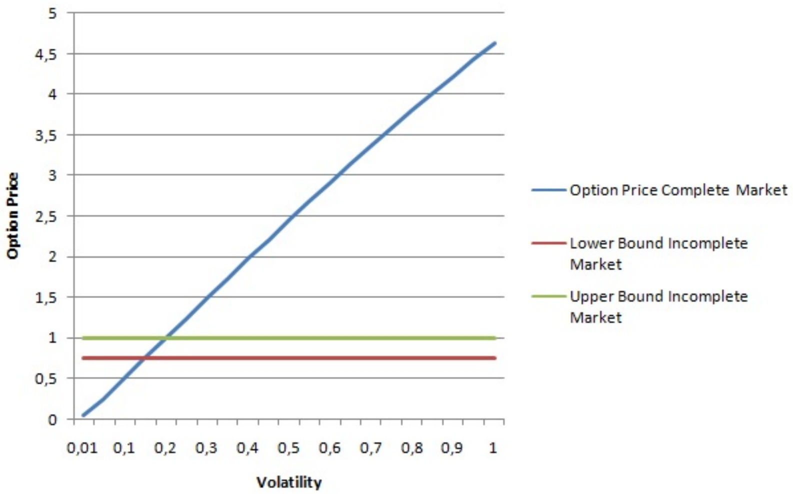


Figure 2.3: Example Call

### 2.4.2 Butterfly Spread

As implied by (2.24), the price of a European call is monotonically increasing in the volatility. This gives rise to the question whether models I and II also agree on arbitrage-free prices of an option under the same conditions if the option price is a decreasing function of the volatility. If they did, the upper arbitrage-free boundary of model II should be intersected by model I's option price at $\sigma_1$, i.e. at the lower of the two volatilities assumed to be possible in model II. The most simple way to investigate this by means of an example would be the construction of a put option; however, testing the observation on a more complex financial derivative provides some more insight in terms of generalisability. For that reason a butterfly spread is used. Consider the following combination of four call options:

- Buy a call with strike $K_1 = 5$
- Sell two calls with strike $K_2 = 10$
- Buy a call with strike $K_3$=15

Moreover, $S_0^1 = S_0^2 = 10$ is again assumed. Butterfly spreads are mainly used by investors who think that large movements in the stock price are unlikely [5], so the lower the volatility and the closer the terminal stock price to $K_2$, the higher the payoff. Thus, if the underlying volatility of the stock price gets larger, the price of the Butterfly Spread decreases. The payoff pattern of this construct is represented in Table 2.1 taken from p.257 in [5].

| Range | Payoff first long | Payoff second long | Payoff two short | **Total** |
|---|---|---|---|---|
| $S_T \leq K_1$ | 0 | 0 | 0 | 0 |
| $K_1 < S_T \leq K_2$ | $S_T - K_1$ | 0 | 0 | $S_T - K_1$ |
| $K_2 < S_T \leq K_3$ | $S_T - K_1$ | 0 | $-2(S_T - K_2)$ | $K_3 - S_T$ |
| $S_T \geq K_3$ | $S_T - K_1$ | $S_T - K_3$ | $-2(S_T - K_2)$ | 0 |

Table 2.1: Payoff Butterfly Spread

In order to find the price range of the Butterfly Spread in model II, the payoffs in the four states can be deduced from Table 2.1 by using the possible stock prices as presented in Figure 2.2. The only difference to the European call example in the previous section in terms of the optimization problem is the change of the objective function in (2.23); the call prices deduced from the Butterfly Spread payoff table are of course inserted. Regarding model I, first of all note that the price of the derivative is not a continuous function of the volatility anymore. For any deterministic volatility $\sigma$, the two possible stock prices at time $1$ are found, classified into Table 2.1 and the price of the Butterfly Spread at time $0$ is then calculated using the risk-neutral probabilities. Figure 2.4 visualizes the situation.

The no-arbitrage boundaries yielded by model II are 3.50283503 and 3.006647731, respectively, which are again plotted as horizontals. Looking closely at the intersection of the prices given by the two models, one can see that it occurs again at $\sigma_1 = 0.15$ and $\sigma_2 = 0.20$. Consistent with the previously formulated expectation, the upper no-arbitrage boundary of model II is intersected by model I's price for the lower volatility. Hence the observation that model I and II agree on arbitrage-free prices if the volatility assumed by model I is in between $\sigma_1$ and $\sigma_2$ seems to become manifest. Before it is tried to find a general explanation for this phenomenon, one more test is done.

### 2.4.3 Double Butterfly Spread

The two previous examples investigated cases where the price of the derivative has been (weakly) monotone in the volatility. It might be interesting to see whether one can make the same observation as before if the price is increasing for some values of the volatility and decreasing for others. To

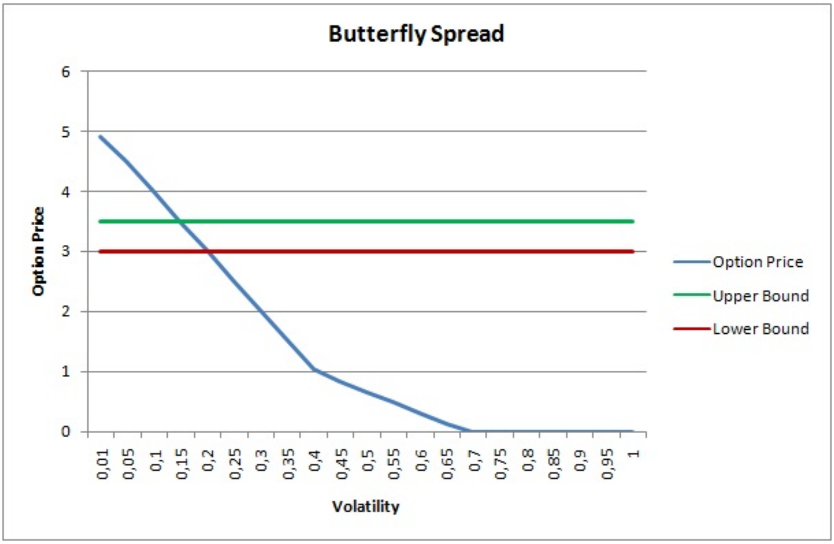


Figure 2.4: Example Butterfly Spread

this end, consider a Double Butterfly Spread: It is simply constructed by putting two normal Butterfly Spreads "next to each other". In the example discussed in the previous subsection, the Butterfly Spread yields the highest payoff if the terminal stock price is as close as possible to the strike of the two shorted calls. On the contrary, if one is uncertain between two values of the final stock price, one can use two Butterfly Spreads in order to target two values. This way two "peak-profits" are created, hence the price of the Double Butterfly Spread is not monotone in the volatility. Assume again that in both models the initial stock price is equal to 10 and consider the following combination of eight calls making up a Double Butterfly Spread:

- Buy a call with strike $K_1 = 6$
- Sell two calls with strike $K_2 = 8$
- Buy two calls with strike $K_3 = 10$
- Sell two calls with strike $K_4 = 12$
- Buy a call with strike $K_5 = 14.$

The two stock prices targeted in this example are obviously 8 and 10. The price function of the Double Butterfly Spread in model I (in terms of the volatility will) be increasing up to a certain value for $\sigma$ and decreasing afterwards. It is certainly possible that the no-arbitrage boundaries given by model II will be intersected more than once.

| Range | 1 long $K_1$ | 2 short $K_2$ | 2 long $K_3$ | 2 short $K_4$ | 1 long $K_5$ | **Total** |
|---|---|---|---|---|---|---|
| $S_T \leq K_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $K_1 < S_T \leq K_2$ | $S_T - K_1$ | 0 | 0 | 0 | 0 | $S_T - K_1$ |
| $K_2 < S_T \leq K_3$ | $S_T - K_1$ | $-2(S_T - K_2)$ | 0 | 0 | 0 | $K_3 - S_T$ |
| $K_3 < S_T \leq K_4$ | $S_T - K_1$ | $-2(S_T - K_2)$ | $2(S_T - K_3)$ | 0 | 0 | $S_T - K_3$ |
| $K_4 < S_T \leq K_5$ | $S_T - K_1$ | $-2(S_T - K_2)$ | $2(S_T - K_3)$ | $-2(S_T - K_4)$ | 0 | $K_5 - S_T$ |
| $K_5 < S_T$ | $S_T - K_1$ | $-2(S_T - K_2)$ | $2(S_T - K_3)$ | $-2(S_T - K_4)$ | $S_T - K_5$ | 0 |

Table 2.2: Payoff Double Butterfly Spread

In order to determine the prices, consider the payoff structure presented in Table 2.2. Again the stock prices given by Figure 2.2 can be classified in the just mentioned table in oder to calculate the payoff of the Double Butterfly Spread in the respective state. These payoffs are then inserted into the optimization problem as objective function. In model I for any volatility $\sigma$ the two possible stock prices are found, the payoffs are determined and the price is calculated using the risk-neutral probabilities. The situation is depicted in Figure 2.5. The no-arbitrage boundaries yielded by model II are 1.497164542 and 1.800679869. Intersections of the prices given by the two models obviously occur at volatilities of 0.15, 0.20 as well as 0.25, where at $\sigma$=0.20 the option price predicted by model I is the highest, thus up to a volatility of 0.20 the option price is increasing and decreasing afterwards. The former observation is now partly supported by the fact that intersections occur at $\sigma_1$ and $\sigma_2$, but there is a larger range of volatilities for which both models agree on arbitrage-free prices. However, the fact that the phenomenon still occurs in a setting where the option price given by model I is neither continuous nor monotonically increasing or decreasing in the volatility gives rise to the conjecture that the observations hold true for the general case, which will be the topic of the next chapter.

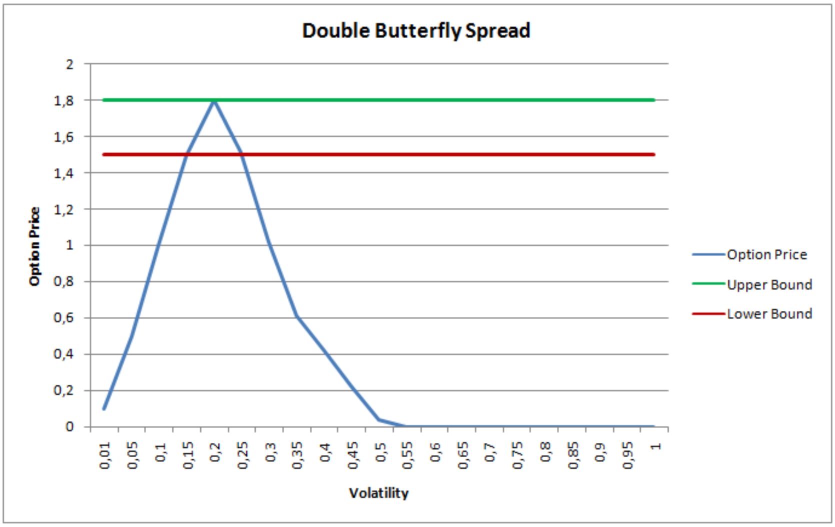


Figure 2.5: Example Double Butterfly Spread

**Internal Summary**

Three examples have confirmed that both models I and II agree on an arbitrage-free price of a derivative security if the underlying volatility $\sigma$ in model I is between $\sigma_1$ and $\sigma_2$. Thereby it did not matter whether the option price was not monotone or not continuous in the volatility in model I.

## 2.5 Generalization and Proof

Before discussing the general case, note that there is an exception: If the strike price $K$ of the option is so high that in none of the possible states there is a positive payoff, both models will certainly yield a price of $0$ regardless whether the volatility in model I is in between the two volatilities assumed in model II. In the remainder of this paper it is assumed that there is at least one positive option payoff at time $1$ in model II. Again denoting the price of a European call option in model I by $C_0^1(\sigma)$, the following theorem is stated:

**Theorem 2.** *For any European Call option, $C_0^1(\sigma^*)$ is an arbitrage-free price in model II if and only if $\sigma^* \in (\sigma_1, \sigma_2)$.*

*Proof.* Assume $C_0^1(\sigma^*)$ is an arbitrage-free call price in model II. Consider the maximal price, i.e. the solution to the maximization problem of (2.23).

Let $\mathbb{X}^2$ be a probabaility measure, that is not necessarily equivalent to $\mathbb{P}^2$, under which the stock price process $S^2$ is a martingale. Under $\mathbb{X}^2$ denote the pseudo-probability leading to state $j$ by $x_j^2$, $0 \leq j \leq 4$. Then the former constraint $q_j^2 > 0$ becomes $x_j^2 \geq 0$ and (2.23) is an instance of a linear optimization problem having a solution that will be determined by means of the Simplex Method. Assume without loss of generality that state 1 belongs to the upward movement of the stock with the underlying volatility $\sigma_1$ and state 2 belongs to the corresponding downward movement. Let states 3 and 4 be defined accordingly. Consider the following variables:

- $x_j^2 \geq 0$: Primal flow variable
- $w_j \geq 0$: Primal slack variable
- $y_j \geq 0$: Dual flow variable
- $z_j \geq 0$: Dual slack variable

After standardizing (2.23) with the modified constraint that the probability leading to state j is not strictly larger than zero, the following initial primal dictionary is obtained:

$$\xi = x_1^2 \cdot c_1^2 + x_2^2 \cdot c_2^2 + x_3^2 \cdot c_3^2 + x_4^2 \cdot c_4^2 \tag{2.27}$$

$$w_1 = 1 - x_1^2 - x_2^2 - x_3^2 - x_4^2 \tag{2.28}$$

$$w_2 = -1 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \tag{2.29}$$

$$w_3 = S_0^2 - s_1^2 \cdot x_1^2 - s_2^2 \cdot x_2^2 - s_3^2 \cdot x_3^2 - s_4^2 \cdot x_4^2 \tag{2.30}$$

$$w_4 = -S_0^2 + s_1^2 \cdot x_1^2 + s_2^2 \cdot x_2^2 + s_3^2 \cdot x_3^2 + s_4^2 \cdot x_4^2 \tag{2.31}$$

(2.32)

Since the slack varibles are partly negative, the primal is infeasible; as the objective function $\xi$ contains positive factors, the dual is infeasible as well [8]. For that reason the primal objective function is modified such that the dual becomes feasible. After solving the articifial problem to optimality, the actual objective function is re-substituted and the algorithm continues. Define the auxiliary function $\xi'$ as follows:

$$\xi' = -x_1^2 - x_2^2 - x_3^2 - x_4^2 \tag{2.33}$$

Then the corresponding dual is given by:

$$-\xi' = -y_1 + y_2 - S_0^2 \cdot y_3 + S_0^2 \cdot y_4 \tag{2.34}$$

$$z_1 = 1 + y_1 - y_2 + s_1^2 \cdot y_3 - s_1^2 \cdot y_4 \tag{2.35}$$
$$z_2 = 1 + y_1 - y_2 + s_2^2 \cdot y_3 - s_2^2 \cdot y_4 \tag{2.36}$$
$$z_3 = 1 + y_1 - y_2 + s_3^2 \cdot y_3 - s_3^2 \cdot y_4 \tag{2.37}$$
$$z_4 = 1 + y_1 - y_2 + s_4^2 \cdot y_3 - s_4^2 \cdot y_4 \tag{2.38}$$
$$\tag{2.39}$$

Choose $y_4$ to enter and $z_3$ to leave the basis:

$$-\xi' = \frac{S_0^2}{s_3^2} + y_1 \cdot (\frac{S_0^2}{s_3^2} - 1) + y_2 \cdot (1 - \frac{S_0^2}{s_3^2}) - z_3 \cdot \frac{S_0^2}{s_3^2} \tag{2.40}$$
$$z_1 = 1 - \frac{s_1^2}{s_3^2} + y_1 \cdot (1 - \frac{s_1^2}{s_3^2}) - y_2 \cdot (1 - \frac{s_1^2}{s_3^2}) + z_3 \cdot \frac{s_1^2}{s_3^2} \tag{2.41}$$
$$z_2 = 1 - \frac{s_2^2}{s_3^2} + y_1 \cdot (1 - \frac{s_2^2}{s_3^2}) - y_2 \cdot (1 - \frac{s_2^2}{s_3^2}) + z_3 \cdot \frac{s_2^2}{s_3^2} \tag{2.42}$$
$$y_4 = \frac{1}{s_3^2} + \frac{y_1}{s_3^2} - \frac{y_2}{s_3^2} + y_3 - \frac{z_3}{s_3^2} \tag{2.43}$$
$$z_4 = 1 - \frac{s_4^2}{s_3^2} + y_1 \cdot (1 - \frac{s_4^2}{s_3^2}) - y_2 \cdot (1 - \frac{s_4^2}{s_3^2}) + z_3 \cdot \frac{s_4^2}{s_3^2} \tag{2.44}$$
$$\tag{2.45}$$

$y_2$ is the only coefficient left in the objective function with a positive coefficient, thus it enters the basis and $z_4$ might leave:

$$-\xi' = 1 + z_3 \cdot A - z_4 \cdot \frac{s_3^2 - S_0^2}{s_3^2 - s_4^2}, \qquad \text{where } A := \frac{s_4^2(s_3^2 - S_0^2)}{(s_3^2 - s_4^2)s_3^2} - \frac{S_0^2}{s_3^2} < 0 \tag{2.46}$$
$$z_1 = 0 + z_3 \cdot B + z_4 \cdot \frac{s_3^2 - s_1^2}{s_3^2 - s_4^2}, \qquad \text{where } B := \frac{s_1^2}{s_3^2} - \frac{s_4^2(s_3^2 - s_1^2)}{s_3^2(s_3^2 - s_4^2)} > 0 \tag{2.47}$$
$$z_2 = 0 + z_3 \cdot C + z_4 \cdot \frac{s_3^2 - s_2^2}{s_3^2 - s_4^2}, \qquad \text{where } C := \frac{s_2^2}{s_3^2} - \frac{s_4^2(s_3^2 - s_2^2)}{s_3^2(s_3^2 - s_4^2)} > 0 \tag{2.48}$$
$$y_4 = 0 - z_3 \cdot (\frac{s_4^2}{s_3^2(s_3^2 - s_4^2)} + \frac{1}{s_3^2}) + z_4 \cdot \frac{1}{s_3^2 - s_4^2} + y_3 \tag{2.49}$$
$$y_2 = 1 + z_3 \cdot \frac{s_4^2}{s_3^2 - s_4^2} - z_4 \cdot \frac{s_3^2}{s_3^2 - s_4^2} + y_1 \tag{2.50}$$

(2.51)

The inequalities can be easily verified. As there is no positive coefficient of a variable in the objective function anymore, the dual, which is feasible, cannot be further increased. The corresponding primal looks as follows:

$$\xi' = -1 - w_2 \tag{2.52}$$

$$w_1 = -w_2 \tag{2.53}$$

$$w_3 = -w_4 \tag{2.54}$$

$$x_3^2 = -A - x_1 \cdot B - x_2 \cdot C + w_4 \cdot (\frac{s_4^2}{s_3^2(s_3^2 - s_4^2)}) + \frac{1}{s_3^2}) - w_2 \cdot \frac{s_4^2}{s_3^2 - s_4^2} \tag{2.55}$$

$$x_4^2 = \frac{s_3^2 - S_0^2}{s_3^2 - s_4^2} - x_1 \cdot \frac{s_3^2 - s_1^2}{s_3^2 - s_4^2} - x_2 \cdot \frac{s_3^2 - s_2^2}{s_3^2 - s_4^2} - w_4 \cdot \frac{1}{s_3^2 - s_4^2} + w_2 \cdot \frac{s_3^2}{s_3^2 - s_4^2} \tag{2.56}$$

Obviously the primal is feasible and cannot be further increased for the auxiliary objective function, thus it is optimal by the *Strong Duality Theorem* [8]. It remains to check whether the optimality also holds for the actual objective function $\xi = x_1^2 \cdot c_1^2 + x_2^2 \cdot c_2^2 + x_3^2 \cdot c_3^2 + x_4^2 \cdot c_4^2$. To this end, the equations for $x_3$ and $x_4$ from the final primal dictionary (2.56) are inserted. By noting the following, some cumbersome calculations can be saved: Since $w_j \geq 0$ by definition, $w_1 = -w_2$ implies $w_1 = w_2 = 0$. The same argument applies to $w_3$ and $w_4$. Thus they cannot increase the objective function. When inserting $x_3^2$ and $x_4^2$ into $\xi$, some simple algebra reveals that their coefficients are not positive, therefore the objective function cannot be increased anymore. Note that the corresponding dual is feasible and can also not be further increased. For that reason the *Strong Duality Theorem* implies that the the optimal solution has been determined [8], proving that in optimality $x_1^2 = x_2^2 = 0$, i.e. the coefficients belonging to the option payoff under $\sigma_1$ are zero. $x_3^2$ and $x_4^2$ are obviously larger than zero, thus the prices given by model I and II must coincide. It follows that $\xi = C_0^1(\sigma_2)$. Going back to the initial problem and reintroducing the martingale measure $\mathbb{Q}^2$ that is equivalent to $\mathbb{P}^2$ imposes the stronger constraint $q_j^2 > 0$. As under the measure $\mathbb{X}^2$ the constraint held with equality in optimum for the probabilities belonging to states 1 and 2, this implies that under $\mathbb{Q}^2$ the value $C_0^1(\sigma_2)$ is not attainable. Since in model I each volatility yields a unique price and if $C_0^1(\sigma^*)$ is arbitrage-free, one can conclude $C_0^1(\sigma_2) > C_0^1(\sigma^*)$. The strict monotonicity of $C$ implies $\sigma_2 > \sigma^*$. By a similar line of reasoning it can be shown that in the minimum $x_3^2 = x_4^2 = 0$ which will lead to the conclusion $\sigma^* > \sigma_1$. Hence, if $C_0^1(\sigma^*)$ is arbitrage-free in model II, it must hold that $\sigma^* \in (\sigma_1, \sigma_2)$.

Conversely, suppose $\sigma^* \in (\sigma_1, \sigma_2)$. The application of the Simplex Method has shown that any arbitrage-free price attainable in model II is strictly larger than $C_0^1(\sigma_1)$ and strictly smaller than $C_0^1(\sigma_2)$. Since $C$ is strictly increasing in $\sigma$, this implies that $C_0^1(\sigma^*)$ is in between these two values. Therefore $C_0^1(\sigma^*)$ is an arbitrage-free price in model II.

□

**Remark**

The calculation with the Simplex Method can easily be verified as follows: The number $-A$ as defined in dictionary (2.51) should correspond to the risk-neutral probability of an upward move in model I when assuming an underlying volatility $\sigma_2$. In other words, $-A$ should be the same as inserting $\sigma_2$ into equation (2.15) (rember the assumption $r_f = 0$). Hence, if the calculation is correct, the following should hold:

$$-A \stackrel{!}{=} \frac{1 - e^{-\sigma_2}}{e^{\sigma_2} - e^{-\sigma_2}} \tag{2.57}$$

This is straightforwardly shown:

## 2.6 The Trinomial Model

In this chapter the Binomial Model is extended to the Trinomial Model. All the assumptions made so far are maintained except the one that for some volatility $\sigma$ there are only two possible values of the share price at time $1$. The Trinomial Model allows the stock price to increase, decrease and to *remain constant*. Lots of authors argue that the Trinomial Model therefore mirrored reality better than the Binomial Model and that it was more appropriate for pricing options. [2] shows that on average the accurary of the Trinomial Model with 5 time steps is comparable to the Binomial Model with 20 time intervals. However, in this paper also the Trinomial Model is just investigated in a two-period economy. The cases of deterministic as well as stochastic volatility are explored in turn and it is possible to build a lot on the theory previously discussed.

**Notational Remark**

The Trinomial Model with deterministic (stochastic) volatility will be referred to as model III (model IV).

### 2.6.1 Deterministic Volatility

Considering the finite filtered probability space $(\Omega^3, \mathbb{F}^3, \mathbb{P}^3)$ and imposing a deterministic volatility in the trinomial setting, there are three possible future states, thus $|\Omega^3| = 3$. Since only two assets are traded in the market (stock and bond), the market is incomplete. For that reason it is not possible to entirely replicate the payoff of a contingent claim in this Trinomial Model, thus no unique (option) price can be calculated and again an interval is obtained. The up and down factors employed for the approximation to the Geometric Brownian Motion might be used again; however, before the familiar optimization problem can be used to model the situation, it is to note that the values of the stock in case it goes up or down cannot be determined by the same up and down factor as in the binomial setting. Using the previous up factor $u = e^{\sigma}$ and assuming $u \cdot d = 1$, [2] shows that the actual probability leading to the state that the stock price remains constant is negative. Consider the general case for some $\lambda$:

$$u = e^{\lambda \cdot \sigma} \tag{2.58}$$

Obviously, in the Binomial Model $\lambda$=1. In order to produce only positive probabilities in the Trinomial Model, $\lambda$ has to be strictly larger than 1 [2]. Many papers simply choose $\lambda = \sqrt{2}$; this being a common choice and since it does not matter too much for the topic at hand, this paper also makes that assumption, thus denote by $u' = e^{\sigma \cdot \sqrt{2}}$ the modified up factor (the assumption $u' \cdot d' = 1$ is kept).
Consider the European call option with strike $K$. In order to determine the no-arbitrage boundaries in this model, the same approach as in the Binomial Model with stochastic volatility is followed. Let $\mathbb{Q}^3$ be an equivalent measure to $\mathbb{P}^3$ under which the stock price process $S^3$ is a martingale. Recall that the *Fundamental Theorem of Asset Pricinig* implies that the $\mathbb{Q}^3$ weighted average of the call's payoff at time $1$ yields an arbitrage free price at time $0$, thus the adjusted optimization problem looks as follows:

$$
\begin{aligned}
\text{max/min} \quad & \sum_{j=1}^{3} q_j^3 \cdot c_j^3 \\
\text{subject to} \quad & \sum_{j=1}^{3} q_j^3 = 1 \\
& \sum_{j=1}^{3} q_j^3 \cdot s_j^3 = S_0^3 \\
& q_j^3 > 0 \;\; \forall j
\end{aligned}
\tag{2.59}
$$

Let the solutions to (2.59) be given by respectively $C_d^3$ and $C_u^3$. The derivation and subsequent discussion of the interval $W$ in section 3.2.1 implies the following corollary for model III:

**Corollary 1.** *For any European call option, any number $C_0^3$ is an arbitrage-free price of the call in model III if and only if $C_0^3 \in (C_d^3, C_u^3)$.*

### 2.6.2 Stochastic Volatility

Similar to model II, in model IV it is supposed that the volatility can take either of the two values $\sigma_1$ and $\sigma_2$ and assume w.l.o.g. $\sigma_1 < \sigma_2$. Let the finite filtered probability space in this case be given by $(\Omega^4, \mathbb{F}^4, \mathbb{P}^4)$. Then, for some $\alpha \in (0,1)$, $\mathbb{P}^4(\sigma = \sigma_1) = \alpha$ and $\mathbb{P}^4(\sigma = \sigma_2) = 1 - \alpha$. It follows that $|\Omega^4|$=6. In order to determine the stock prices in case of upward and downward movements, of course the same up and down factors are used as in the case of deterministic volatility in the Trinomial Model. Defining $\mathbb{Q}^4$ as usual, the following optimization problem can now be stated immediately:

$$
\begin{aligned}
\text{max/min} \quad & \sum_{j=1}^{6} q_j^4 \cdot c_j^4 \\
\text{subject to} \quad & \sum_{j=1}^{6} q_j^4 = 1 \\
& \sum_{j=1}^{6} q_j^4 \cdot s_j^4 = S_0^4 \\
& q_j^4 > 0 \;\; \forall j
\end{aligned}
\tag{2.60}
$$

Denote here the no-arbitrage boundaries by $C_d^4$ as well as $C_u^4$. In a similar spirit as for model III, the following corollary regarding the no-arbitrage interval in model IV can be stated:

**Corollary 2.** *For any European call option, any number $C_0^4$ is an arbitrage-free price of the call in model IV if and only if $C_0^4 \in (C_d^4, C_u^4)$.*

## 2.7 Numerical Examples II

In this chapter some numerical examples of the results of models III and IV are provided. During the comparison of model I and II, the output has been one line and one interval. In the case at hand, the results will of course be three intervals, namely two intervals yielded by model III with deterministic volatiltiies $\sigma_1$ and $\sigma_2$ and one interval given by model IV with the two volatilities being stochastic. Particular attention is then paid to the relation of the boundaries. To this end, a European Call, a Butterfly Spread and a Double Butterfly Spread are again considered.
Throughout all the following examples it is once more assumed that the initial stock price in both models is equal to 10, $\sigma_1 = 0.15$ and $\sigma_2 = 0.20$. The solutions to the optimization problems are again determined with the help of Aimms 3.12.

### 2.7.1 Description

#### European Call

Obtaining no-arbitrage boundaries for a European call in models III and IV is straightforwardly done. Suppose that the strike price of the call is given by $K = 10$. Figure 2.6 shows the possible stock prices and the corresponding call option payoff in the braces below. In order to determine the two no-arbitrage intervals from model III, the "upper part" of Figure 2.6 is put into problem (2.59) and the same is subsequently done with the "lower part". To obtain the no-arbitrage boundaries of model IV, all the values shown in Figure 2.6 are inserted into problem (2.60). The results are given in Table 2.3. The first row displays the no arbitrage boundaries yielded by model III for a deterministic volatility of 0.15, the second one shows the corresponding results for a volatility of 0.20 and the final row contains the boundaries given by model IV assuming a stochastic volatility. On a first glance the lower bounds seem to be unrealistically low, but extracting the corresponding solutions from Aimms and checking the results manually showed that all the constraints are met and the calculated value is correct.

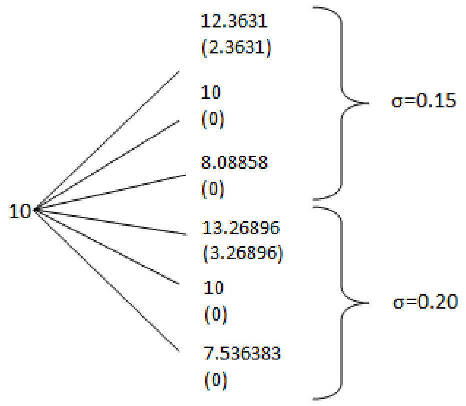


Figure 2.6: Six States

The upper bound of model III with $\sigma$=0.20 is the highest among all the upper bounds. Since the call is increasing in the volatility, it makes sense that this upper bound is larger than the one obtained assuming $\sigma$=0.15. The upper bound yielded by model IV lies in between. Likewise it is reasonable that the lower bound for $\sigma$=0.15 is the lowest among all lower bounds though they are all relatively close to each other.

| Volatility Regime | Lower Bound | Upper Bound |
|---|---|---|
| $\sigma = 0.15$ | $1.0566912 \cdot 10^{-6}$ | 1.05669 |
| $\sigma = 0.20$ | $3.269 \cdot 10^{-6}$ | 1.40485999 |
| *Stochastic* | $4.375 \cdot 10^{-6}$ | 1.4048579 |

Table 2.3: Boundaries Call

### Butterfly Spread

In this part the models are applied to pricing a Butterfly Spread that has the same setup, two long calls and two short calls with the same strike prices, as is the one described in section 4.2. In order to determine the payoff at time $1$, the stock prices shown in Figure 2.6 can be classified into Table 2.1 and then the values are easily deducible. Adapting the objective functions in (2.59) and (2.60), one obtains the upper and lower bounds for the price of the Butterfly Spread shown in Table 2.4. Recall that the Butterfly Spread's price is decreasing in the volatility, therefore it does not come as a surprise that the upper bound for $\sigma$=0.15 is the highest among all upper bounds. This also explains why the lower bound obtained by imposing a deterministic volatility of $\sigma$=0.20 yields the lowest among all lower bounds where the one obtained by using model IV is slightly higher. In the previous example when the option price has been increasing in the volatility, all the lower bounds were clustering; in this case all the upper bounds are very close to each other.

| Volatility Regime | Lower Bound | Upper Bound |
|---|---|---|
| $\sigma = 0.15$ | 2.888618 | 4.9999957 |
| $\sigma = 0.20$ | 2.19028 | 4.99999 |
| *Stochastic* | 2.190284 | 4.999991 |

Table 2.4: Boundaries Butterfly Spread

### Double Butterfly Spread

Finally the two models are used to determine the no-arbitrage boundaries of a Double Butterfly Spread and the same construction is used as in section 4.3. To this end, the stock values displayed in Figure 2.6 are compared to the payoff pattern described by Table 2.2 and inserted into problems (2.59) as well as (2.60). Table 2.5 summarizes the results (the lower bounds have been verified again). The upper bound for a volatility of $\sigma$=0.15 is the highest although the corresponding value of model IV is only slightly lower. The lower bounds are all clustering slightly above zero; here the value for the deterministic volatility of $\sigma_2$ is the lowest among all lower bounds.

| Volatility Regime | Lower Bound | Upper Bound |
|---|---|---|
| $\sigma = 0.15$ | $1.788654 \cdot 10^{-6}$ | 1.7886519 |
| $\sigma = 0.20$ | $1.190277 \cdot 10^{-6}$ | 1.190276 |
| *Stochastic* | $6.599678 \cdot 10^{-6}$ | 1.7886488 |

Table 2.5: Boundaries Double Butterfly Spread

## 2.7.2 Observation

When discussing the exampels for models I and II, the observation regarding the intersection at the volatilities $\sigma_1$ and $\sigma_2$ was easily made. In this case finding a general pattern is somehow more difficult, but looking closely at Tables 2.3, 2.4 and 2.5, one can see that the lower bound obtained by using model IV is always *above* one of the lower bounds yielded by model III. To be more precise, in the example dealing with the call option, the smallest lower bound is given by model III for $\sigma = 0.15$ and model IV's lower bound is larger. In the second example, the minimal lower bound is of course given by model III for $\sigma = 0.20$ and model IV's lower bound is again larger. In the third example, both lower bounds given by model III are smaller than the one given by model IV. Considering the upper bounds, the same observation can be made reversed: In the first example, model III gives the highest upper bound for $\sigma$=0.20 and the corresponding value of model IV is slightly lower. In case of the Butterfly Spread, the highest value is given by III for a volatility of 0.15 and model IV's upper bound is a bit below. Regarding the Double Butterfly Spread, model IV can also not reach the largest upper bound. As this phenomenon occurs throughout examples where the derivative security's price has been increasing, decreasing and not monotone in the volatility, this gives rise to the conjecture that model IV's no-arbitrage boundaries are less extreme than the boundaries yielded by model III when the same values for the volatilities are assumed. The next chapter sheds some further light on this question.

## 2.8 General Case European Call

In the general discussion, attention is restricted to the no-arbitrage boundaries obtained when pricing a European call. Similar to chapter 5, a martingale measure that is not necessarily equivalent to the actual probability measure is used in order to to find an analytical solution. Thus, let $\mathbb{X}^3$ be defined such that the stock prices process in model III is a martingale under this measure and define $\mathbb{X}^4$ for model IV accordingly. Recall that under a martingale measure that is not necessarily equivalent to the original probability measure, the probability leading to state j, $0 \leq j \leq 6$, is not forced to be strictly larger than zero.
Consider model IV in the following. Assume w.l.o.g. that state 1 refers to the stock price going up, state 2 to no change and state 3 to a decrease under $\sigma_1$; let the other states for $\sigma_2$ be labeled in a similar manner. If it was possible to show that in the maximum the probabilities leading to states 1,2 and 3 were equal to zero, this would prove that model IV and model III would yield the same value for a deterministic volatility of $\sigma_2$. The Simplex Algorithm is used to find the maximum. To this end, define the following variables:

- $x_j^4 \geq 0$: Primal flow variable
- $w_j \geq 0$: Primal slack variable
- $y_j \geq 0$: Dual flow variable

- $z_j \geq 0$: Dual slack variable

After standardizing, the following primal dictionary is obtained:

$$\xi = c_1^4 \cdot x_1^4 + c_2^4 \cdot x_2^4 + c_3^4 \cdot x_3^4 + c_4^4 \cdot x_4^4 + c_5^4 \cdot x_5^4 + c_6^4 \cdot x_6^4 \quad (2.61)$$
$$w_1 = 1 - x_1^4 - x_2^4 - x_3^4 - x_4^4 - x_5^4 - x_6^4 \quad (2.62)$$
$$w_2 = -1 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 \quad (2.63)$$
$$w_3 = S_0^4 - s_1^4 \cdot x_1^4 - s_2^4 \cdot x_2^4 - s_3^4 \cdot x_3^4 - s_4^4 \cdot x_4^4 - s_5^4 \cdot x_5^4 - s_6^4 \cdot x_6^4 \quad (2.64)$$
$$w_4 = -S_0^4 + s_1^4 \cdot x_1^4 + s_2^4 \cdot x_2^4 + s_3^4 \cdot x_3^4 + s_4^4 \cdot x_4^4 + s_5^4 \cdot x_5^4 + s_6^4 \cdot x_6^4 \quad (2.65)$$
$$(2.66)$$

The slack variables are obviously partly negative, thus the primal is infeasible. Taking the corresponding dual, the same observation would be made since the primal objective function contains positive values. To overcome this problem, an auxiliary objective function is defined, the problem is then solved to optimatility and later on the initial objective function has to be re-substituted. Suppose the temporary objective function is given by:

$$\xi' = -x_1^4 - x_2^4 - x_3^4 - x_4^4 - x_5^4 - x_6^4 \quad (2.67)$$

Then the following dual dictonary is of course feasible:

$$-\xi' = -y_1 + y_2 - S_0^4 \cdot y_3 + S_0^4 \cdot y_4 \quad (2.68)$$
$$z_1 = 1 + y_1 - y_2 + s_1^4 \cdot y_3 - s_1^4 \cdot y_4 \quad (2.69)$$
$$z_2 = 1 + y_1 - y_2 + s_2^4 \cdot y_3 - s_2^4 \cdot y_4 \quad (2.70)$$
$$z_3 = 1 + y_1 - y_2 + s_3^4 \cdot y_3 - s_3^4 \cdot y_4 \quad (2.71)$$
$$z_4 = 1 + y_1 - y_2 + s_4^4 \cdot y_3 - s_4^4 \cdot y_4 \quad (2.72)$$
$$z_5 = 1 + y_1 - y_2 + s_5^4 \cdot y_3 - s_5^4 \cdot y_4 \quad (2.73)$$
$$z_6 = 1 + y_1 - y_2 + s_6^4 \cdot y_3 - s_6^4 \cdot y_4 \quad (2.74)$$
$$(2.75)$$

$y_4$ enters the basis, then $z_4$ may leave:

$$-\xi' = \frac{S_0^4}{s_4^4} + y_1(\frac{S_0^4}{s_4^4} - 1) + y_2(1 - \frac{S_0^4}{s_4^4}) - z_4 \cdot \frac{S_0^4}{s_4^4} \quad (2.76)$$

$$z_1 = 1 - \frac{s_1^4}{s_4^4} + y_1(1 - \frac{s_1^4}{s_4^4}) + y_2(\frac{s_1^4}{s_4^4} - 1) + z_4 \cdot \frac{s_1^4}{s_4^4} \tag{2.77}$$

$$z_2 = 1 - \frac{s_2^4}{s_4^4} + y_1(1 - \frac{s_2^4}{s_4^4}) + y_2(\frac{s_2^4}{s_4^4} - 1) + z_4 \cdot \frac{s_2^4}{s_4^4} \tag{2.78}$$

$$z_3 = 1 - \frac{s_3^4}{s_4^4} + y_1(1 - \frac{s_3^4}{s_4^4}) + y_2(\frac{s_3^4}{s_4^4} - 1) + z_4 \cdot \frac{s_3^4}{s_4^4} \tag{2.79}$$

$$y_4 = \frac{1}{s_4^4} + \frac{y_1}{s_4^4} - \frac{y_2}{s_4^4} + y_3 - \frac{z_4}{s_4^4} \tag{2.80}$$

$$z_5 = 1 - \frac{s_5^4}{s_4^4} + y_1(1 - \frac{s_5^4}{s_4^4}) + y_2(\frac{s_5^4}{s_4^4} - 1) + z_4 \cdot \frac{s_5^4}{s_4^4} \tag{2.81}$$

$$z_6 = 1 - \frac{s_6^4}{s_4^4} + y_1(1 - \frac{s_6^4}{s_4^4}) + y_2(\frac{s_6^4}{s_4^4} - 1) + z_4 \cdot \frac{s_6^4}{s_4^4} \tag{2.82}$$

$$\tag{2.83}$$

Since $y_2$ is the only variable with a positive coefficient left in the objective function, it enters the basis and $z_6$ might leave:

$$-\xi' = 1 + z_4 \cdot A - z_6 \cdot \frac{s_4^4 - S_0^4}{s_4^4 - s_6^4} \qquad \text{with } A := \frac{s_6^4(s_4^4 - S_0^4)}{s_4^4(s_4^4 - s_6^4)} - \frac{S_0^4}{s_4^4} < 0 \tag{2.84}$$

$$z_1 = z_4 \cdot B - z_6 \cdot \frac{s_1^4 - s_4^4}{s_4^4 - s_6^4} \qquad \text{with } B := \frac{s_6^4(s_1^4 - s_4^4)}{s_4^4(s_4^4 - s_6^4)} + \frac{s_1^4}{s_4^4} > 0 \tag{2.85}$$

$$z_2 = z_4 \cdot C - z_6 \cdot \frac{s_2^4 - s_4^4}{s_4^4 - s_6^4} \qquad \text{with } C := \frac{s_6^4(s_2^4 - s_4^4)}{s_4^4(s_4^4 - s_6^4)} + \frac{s_2^4}{s_4^4} > 0 \tag{2.86}$$

$$z_3 = z_4 \cdot D - z_6 \cdot \frac{s_3^4 - s_4^4}{s_4^4 - s_6^4} \qquad \text{with } D := \frac{s_6^4(s_3^4 - s_4^4)}{s_4^4(s_4^4 - s_6^4)} + \frac{s_3^4}{s_4^4} > 0 \tag{2.87}$$

$$y_4 = -z_4 \cdot E + z_6 \cdot \frac{1}{s_4^4 - s_6^4} + y_3 \qquad \text{with} E := \frac{s_6^4}{s_4^4(s_4^4 - s_6^4)} + \frac{1}{s_4^4} > 0 \tag{2.88}$$

$$z_5 = z_4 \cdot F - z_6 \cdot \frac{s_5^4 - s_4^4}{s_4^4 - s_6^4} \qquad \text{with } F := \frac{s_6^4(s_5^4 - s_4^4)}{s_4^4(s_4^4 - s_6^4)} + \frac{s_5^4}{s_4^4} > 0 \tag{2.89}$$

$$y_2 = 1 + y_1 + z_4 \cdot \frac{s_6^4}{s_4^4 - s_6^4} - z_6 \cdot \frac{s_4^4}{s_4^4 - s_6^4} \tag{2.90}$$

The inequalities can be easily verified. Note that there is no variable with positive coefficient in the above objective function anymore. Since the dictionary is feasible, it has been solved as far as possible. The next thing to do is the translation back into the primal:

$$\xi' = -1 - w_2 \tag{2.91}$$

$$x_4^4 = -A - x_1^4 \cdot B - x_2^4 \cdot C - x_3^4 \cdot D + w_4 \cdot E - x_5^4 \cdot F - w_2 \cdot \frac{s_6^4}{s_4^4 - s_6^4} \tag{2.92}$$

$$x_6^4 = \frac{s_4^4 - S_0^4}{s_4^4 - s_6^4} + x_1^4 \cdot \frac{s_1^4 - s_4^4}{s_4^4 - s_6^4} + x_2^4 \cdot \frac{s_2^4 - s_4^4}{s_4^4 - s_6^4}$$

$$+ x_3^4 \cdot \frac{s_3^4 - s_4^4}{s_4^4 - s_6^4} + w_4 \cdot \frac{1}{s_4^4 - s_6^4} + x_5^4 \cdot \frac{s_5^4 - s_4^4}{s_4^4 - s_6^4} + w_2 \cdot \frac{s_4^4}{s_4^4 - s_6^4} \tag{2.93}$$

$$w_1 = -w_2 \tag{2.94}$$

$$w_3 = -w_4 \tag{2.95}$$

$$\tag{2.96}$$

The above dictionary is feasible and $\xi'$ cannot be further increased. The *Strong Duality Theorem* implies that the problem has been solved to optimality [8]. Going back to the initial objective function $\xi = c_1^4 \cdot x_1^4 + c_2^4 \cdot x_2^4 + c_3^4 \cdot x_3^4 + c_4^4 \cdot x_4^4 + c_5^4 \cdot x_5^4 + c_6^4 \cdot x_6^4$, the equations describing $x_4^4$ and $x_6^4$ have to be inserted. In the above final primal, all the slacks are equal to zero, thus they cannot increase the initial objective function anymore. Moreover, it can be easily calculated that the coefficients of $x_1^4, x_2^4, x_3^4$ and $x_5^4$ are not positive in $\xi$ when $x_4^4$ and $x_6^4$ are inserted, hence $\xi$ cannot be further increased. Recognizing that the corresponding dual is feasible and that it can also not be further increased, the maximum for the initial problem has been found (*Strong Duality Theorem*).

Note that the minimum value can be found by a completely similar line of reasoning, thus it is not explicitly reproduced here. However, the result one obtains is that there is a solution to the minimization problem of model IV such that $x_4^4 = x_5^4 = x_6^4 = 0$. Therefore it agrees with model III on the minimal value if the latter assumes a deterministic volatility of $\sigma_1$. The calculations just done prove the following theorem which is going to summarize this chapter.

**Theorem 3.** *The following two statements hold true:*

- *The infimum arbitrage-free price of a call in model III with deterministic volatility $\sigma_1$ is equal to the infimum arbitrage-free of a call in model IV with stochastic volatilities $\sigma_1$ and $\sigma_2$*

- *The supremum arbitrage-free price of a call in model III with deterministic volatility $\sigma_2$ is equal to the supremum arbitrage-free of a call in model IV with stochastic volatilities $\sigma_1$ and $\sigma_2$*

## 2.9 Conclusion

This paper has been devoted to the theory of pricing financial deriatives in cases of deterministic as well as stochastic volatility of the underlying. Firstly, the Binomial Model with deterministic volatility has been introduced and a unique price could be obtained. After allowing for a second value of the volatility, the market setting became incomplete and the description of a derivative's price has taken the form of an interval. Some numerical examples have been discussed from which a general result regarding the valuation of a European call option has been deduced and subsequently formally proved. Then attention has been paid to the Trinomial Model, first in case of deterministic and later under a stochastic volatility. In both cases only intervals for the derivatives' prices could be obtained. For illustrational purposes some examples have been presented; in this case it was also possible to detect a general pattern: The no-arbitrage boundaries in the Trinomial Model with deterministic volatility always seemed to be more extreme than the corresponding boundaries when the volatility has been assumed to be stochastic. A proof that supremum and infimum of the Trinomial Model assuming respectively deterministic and stochastic volatility conincide has been provided. Presumably it is in general the case that under an equivalent martingale measure the boundaries in model IV are less extreme than in model III. Investigating and proving this in a formal way is a good point for conducting further research.

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